Statistica & Applicazioni - Vol. VII, n. 2, 2009, pp. 135-147

MOVING EXTREME RANKED SET SAMPLING FOR SIMPLE LINEAR REGRESSION

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SUMMARY

The moving extreme ranked set sampling, introduced by Alodat and Al-Saleh (2001), is a modification of the well known ranked set sampling approach that was proposed by McIntyre (1952). In this paper, we suggest new estimators for the simple linear regression parameters under the moving extreme ranked set sampling scheme. Moreover, we show that the proposed estimators are more efficient than their counterparts using the simple random sampling approach. We illustrate our ideas and thoughts via simulation and data analysis and conduct a comparison between our approach and the traditional ones.

Keywords: *Moving ranked set sampling, Ranked set sampling, Simple linear regression.*

1. INTRODUCTION

In his pioneer and distinguished article, McIntyre (1952) introduced an ingenious method for selecting a sample that is considered to be more informative than a simple random sample for estimating the mean of the population of interest. The McIntyre's sampling scheme is called later the Ranked Set Sampling (RSS). The new idea was studied extensively in the literature and many theoretical properties as well as computational features have been introduced over the last five decades (for more details, see Takahasi and Wakitmoto, 1968; Muttlak, 1997; Chen, Bai and Sinha, 2003 and Alodat, Al-Rawwash and Nawajah, 2010). Following the footsteps of many researchers, we denote McIntyre's sampling scheme and the sample obtained accordingly by an RSS. The basic idea behind selecting an RSS can be described as follows: Select m random samples each of size m. Then, using visual inspection, rank the units within each sample with respect to the variable of interest. Accordingly we select, for actual measurement, the i^{th} smallest unit from the i^{th} sample, $i = 1, 2, ..., m$, comprising a total of m measured units, one from each sample. Eventually, we could repeat this procedure r times to obtain an RSS of size mr measurements.

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Numerous articles have pointed out that the efficiency of an RSS is affected by the set size (m) and the ranking errors. In fact, the larger the set size the larger the efficiency of the RSS, also it has been reported that the larger the set size the more the difficulties in visual ranking and therefore the larger the ranking error (Al-Saleh and Al-Omari, 2002). For this, several authors have modified the RSS to reduce the ranking error and to make visual ranking tractable by experimenters (Samawi, Ahmed and Abu-Dayyeh, 1996; Muttlak, 1997; Alodat and Al-Saleh, 2001; Muttlak, 2003 and Alodat and Al-Sagheer, 2007). Alodat and Al-Saleh (2001) introduced the concept of varied set size RSS, which is known as Moving Extremes Ranked Set Sampling (MERSS). They showed that the new modification produces more efficient estimators for location and scale parameters compared to the RSS scheme. To elaborate more on the MERSS, we outline this procedure in the following steps:

1. Select $m-1$ simple random samples of size $2, 3, ..., m$, respectively.
2. For the j^{th} sample, $j = 2, 3, ..., m$, we quantify the j^{th} order statistic using a visual inspection.
3. Repeat the steps 1 and *2,* but by quantifying the minimum from the j^{th} sample.
4. The procedure (1)-(3) could be repeated several times to obtain a larger sample size.

Al-Saleh and Al-Hadrami (2003) suggested using the maximum likelihood estimator (MLE) for the symmetry parameter under MERSS and showed that it is more efficient than its Simple Random Sample (SRS) counterpart. Alodat, Al-Rawwash and Nawajah (2009) adopted the simple linear regression model and used the Median Ranked Set Sample (MRSS) to produce estimators and confidence intervals for the normal distribution parameters. Later, Alodat *et al.* (2010) developed the mathematical and statistical properties of the parameters estimators of the simple linear regression model using MRSS. In this paper, we develop and evaluate an estimation method for the simple linear regression parameters assuming the MERSS strategy. Also, we discuss the statistical properties of these estimators including the consistency and efficiency compared to the SRS approach. The article is organized as follows. In Section 2, we introduce the model as well as some necessary basic definitions. Later, we derive the estimators of the simple linear model parameters in Section 3 and we elaborate more on their statistical properties. In Section 4, we discuss the estimation of the dispersion parameter using MERSS. The new proposed method is illustrated in Section 5 via an example that relates the weights of newborn babies to the bilirubin level in their blood. We conclude this article in Section 6.

2. MERSS FOR SIMPLE LINEAR REGRESSION

Consider the simple linear regression model:

$$Y_j = \beta_o + \beta_1 x_j + \varepsilon_j,\ j = 2, 3, ..., m.$$

where $\varepsilon_j's$ are independent and identically distributed $N(0, \sigma^2)$. In this article, we assume that the variable X can be measured with low cost, while measuring the response variable Y is expensive. To discuss and analyze the simple linear regression model via MERSS, we apply the following sampling strategy:

1. Let $x_2, x_3, ..., x_m$ be different values of the variable X which are chosen prior to the experiment.
2. Repeat the experiment $2j$ times for each value x_j. $j = 2, 3, ..., m$, and let $Y_{11}, Y_{12} ..., Y_{1j}$, $Y_{21}, Y_{22}, ..., Y_{2j}$ be the corresponding values of Y, respectively.
3. Let $Y_{1:j} = min(Y_{11}, Y_{12}, ..., Y_{1j})$ and $Y_{j:j} = max(Y_{21}, Y_{22}, ..., Y_{2j})$ for $j = 2, 3, ..., m$. Here $Y_{1:j}$'s and $Y_{j:j}$'s, $j = 2, 3, ...m$, represent a MERSS as described in the previous section.

   Based on this, we adopt the following simple linear models

$$Y_{1:j} = \beta_0 + \beta_1 x_j + \varepsilon_{1:j},$$

$$Y_{j:j} = \beta_0 + \beta_1 x_j + \varepsilon_{j:j},$$

for $j = 2, 3, ..., m$, $\varepsilon_{1:j} = min\{\varepsilon_{11}, ..., \varepsilon_{1j}\}$ and $\varepsilon_{j:j} = max\{\varepsilon_{21}, ..., \varepsilon_{2j}\}$ where ε_{1j} 's are the corresponding errors. It can be noted that the error terms ε_{1j} and ε_{jj} in the last two equations have asymmetric distributions. For these reasons, we intend to rewrite the previous model so that we get a simple linear regression model with symmetric distribution errors. In other words, we end up using the following simple linear regression model

$$Y_j^* = \beta_0 + \beta_1 x_j + \varepsilon_j^*, \quad j = 2, 3, ..., m,$$

where

$$Y_j^* = \frac{1}{2}(Y_{1:j} + Y_{j:j}), \ \varepsilon_j^* = \frac{1}{2}(\varepsilon_{1:j} + \varepsilon_{j:j}), \ \overline{Y^*} =$$

$$= \frac{1}{(m-1)} \sum\nolimits_{j=2}^{m} Y_j^* \text{ and } \overline{\chi} = \frac{1}{(m-1)} \sum\nolimits_{j=2}^{m} \chi_j.$$

It can be noted that the random errors ε_j^* *'s* have zero mean and non-constant variance i.e., $E(\varepsilon_j^*) = 0$ and

$$Var(\varepsilon_j^*) = \frac{1}{4} Var(\varepsilon_{1:j} + \varepsilon_{j:j}),$$

$$= \frac{1}{2} Var(\varepsilon_{j:j}) = \frac{\sigma^2}{2} D_j,$$

where

$$D_j = j \int_{-\infty}^{\infty} z^2 \Phi^{j-1}(z) \varphi(z) dz - \left(j \int_{-\infty}^{\infty} z \Phi^{j-1}(z) \varphi(z) dz \right)^2$$

Such that $\Phi(z)$ and $\varphi(z)$ are the cumulative distribution and the density functions of the standard normal distribution. Arnold, Balakrishnan and Nagaraja (1992) reported the values of D_j for different values of j. In the next two sections and based on the data (x_j, Y_j^*), $j = 2, 3 \ldots, m$ we plan to present the framework of our estimation strategy and discuss the theoretical aspects of the estimates of the regression parameters β_0, and β_1 as well as the dispersion parameter σ.

3. ESTIMATING β_0 AND β_1

A question naturally arises as to whether or not one can guess the proper estimation method based on the information in hand. Had we know the answer in advance, the problem in many situations will be solved. Despite the many estimation methodologies available in the rich menu of statistical literature, we focus our attention on the weighted least squares (WLS) method to carry out the estimation process of the parameters β_0 and β_1. Motivated by the idea that regressing Y_j^* one x_j has non-constant variance and the flexibility of the WLS method, we explain in the sequel the details of our proposed estimators of β_0 and β_1. In fact, the proposed weighted least squares estimators are obtained by minimizing the following weighted sum of squared errors with respect to β_0 and β_1, simultaneously

$$f(\beta_0, \beta_1) = \sum_{j=2}^{m} w_j (Y_j^* - \beta_0 - \beta_1 x_j)^2,$$

To accomplish this, we obtain the partial derivatives with respect to β_0 and β_1 as follows

$$\frac{\partial f(\beta_0, \beta_2)}{\partial \beta_0} = -2 \sum_{j=2}^{m} w_j (Y_j^* - \beta_0 - \beta_1 x_j)$$

and

$$\frac{\partial f(\beta_0, \beta_1)}{\partial \beta_1} = -2 \sum_{j=2}^{m} w_j x_j (Y_j^* - \beta_0 - \beta_1 x_j).$$

Setting the derivatives equal to zero and straightforward simplifications yield the following two equations

$$\beta_0 \sum_{j=2}^{m} w_j + \beta_1 \sum_{j=2}^{m} w_j x_j = \sum_{j=2}^{m} w_j Y_j^* \tag{1}$$

$$\beta_0 \sum_{j=2}^{m} w_j x_j + \beta_1 \sum_{j=2}^{m} w_j x_j^2 = \sum_{j=2}^{m} w_j x_j Y_j^* \tag{2}$$

Therefore, the weighted least squares estimators of β_0 and β_1 are obtained accordingly by solving equations (1) and (2) simultaneously which leads to

$$\hat{\beta}_{1w} = \frac{\sum_{j=2}^{m} w_j x_j Y_j^* - \frac{\left(\sum_{j=2}^{m} w_j x_j\right)\left(\sum_{j=2}^{m} w_j Y_j^*\right)}{\sum_{j=z}^{m} w_j}}{\sum_{j=z}^{m} w_j x_j^2 - \frac{\left(\sum_{j=2}^{m} w_j x_j\right)^2}{\sum_{j=2}^{m} w_j}}$$

and

$$\hat{\beta}_{ow} = \frac{\sum_{j=2}^{m} w_j Y_j^* - \hat{\beta}_{1w} \sum_{j=2}^{m} w_j x_j}{\sum_{j=2}^{m} w_j}$$

where $w_j = \dfrac{1}{D_j}$.

THEOREM 1

Based on the assumptions of Section 2 and using the MERSS and the WLS method, we have

1. $\hat{\beta}_{0w}$ and $\hat{\beta}_{1w}$ are unbiased estimators for β_0 and β_1 respectively.

2. The variance of $\hat{\beta}_{1w}$ is given by

$$Var(\hat{\beta}_{1w}) = \frac{\sigma^2}{2\sum_{j=2}^{m} w_j \left(x_j - \frac{\sum_{i=2}^{m} w_i x_i}{\sum_{i=2}^{m} w_i}\right)^2}$$

3. The variance of $\hat{\beta}_{0w}$ is

$$Var(\hat{\beta}_{ow}) = \frac{\sigma^2}{2}\left[\frac{1}{\sum_{j=2}^{m} w_j} + \frac{\left(\sum_{j=2}^{m} P_j x_j\right)^2}{\sum_{k=2}^{m} w_k \left(x_k - \frac{\sum_{i=2}^{m} w_i x_i}{\sum_{i=2}^{m} w_i}\right)^2}\right]$$

where $P_j = \dfrac{w_j}{\sum_{j=2}^{m} w_j}$.

PROOF:

1) First of all, we may notice that $\hat{\beta}_{1w}$ can be written as

$$\hat{\beta}_{1w} = \frac{\sum_{j=2}^{m} w_j \left(x_j - \frac{\sum_{i=2}^{m} w_i x_i}{\sum_{i=2}^{m} w_i}\right) Y_j^*}{\sum_{j=2}^{m} w_j \left(x_j - \frac{\sum_{i=2}^{m} w_i x_i}{\sum_{i=2}^{m} w_i}\right)^2} \tag{3}$$

Consequently, we take the expectation of both sides of (3) to get

$$E(\hat{\beta}_{1w}) = \frac{\sum_{j=2}^{m} w_j\left(x_j - \frac{\sum_{i=2}^{m} w_i x_i}{\sum_{i=2}^{m} w_i}\right)EY_j^*}{\sum_{j=2}^{m} w_j\left(x_j - \frac{\sum_{i=2}^{m} w_i x_i}{\sum_{i=2}^{m} w_i}\right)^2},$$

$$= \frac{\sum_{j=2}^{m} w_j\left(x_j - \frac{\sum_{i=2}^{m} w_i x_i}{\sum_{i=2}^{m} w_i}\right)(\beta_0 + \beta_1 x_j)}{\sum_{j=2}^{m} w_j\left(x_j - \frac{\sum_{i=2}^{m} w_i x_i}{\sum_{i=2}^{m} w_i}\right)^2},$$

$$= \beta_1.$$

Similarly,

$$E(\hat{\beta}_{0w}) = \frac{\sum_{j=2}^{m} w_j E(Y_j^*) - E(\hat{\beta}_{1w})\sum_{j=2}^{m} w_j x_j}{\sum_{j=2}^{m} w_j}$$

$$= \frac{\sum_{j=2}^{m} w_j(\beta_0 + \beta_1 x_j) - \beta_1 \sum_{j=2}^{m} w_j x_j}{\sum_{j=2}^{m} w_j}$$

$$= \beta_0.$$

2) To determine the efficiencies of $\hat{\beta}_{0w}$ and $\hat{\beta}_{1w}$ with respect to their counterparts using SRS, we need to find the variances of $\hat{\beta}_{0w}$ and $\hat{\beta}_{1w}$. To carry out this mission, we obtain the variance of $\hat{\beta}_{1w}$ in (3) as follows

$$Var(\hat{\beta}_{1w}) = \frac{\sum_{j=2}^{m} w_j^2\left(x_j - \frac{\sum_{i=2}^{m} w_i x_i}{\sum_{i=2}^{m} w_i}\right)^2 Var(Y_j^*)}{\left(\sum_{j=2}^{m} w_j\left(x_j - \frac{\sum_{i=2}^{m} w_i x_i}{\sum_{i=2}^{m} w_i}\right)^2\right)^2},$$

$$= \frac{\sum_{j=2}^{m} w_j^2\left(x_j - \frac{\sum_{i=2}^{m} w_i x_i}{\sum_{i=2}^{m} w_i}\right)^2 \frac{\sigma^2}{2} D_j}{\left(\sum_{j=2}^{m} w_j\left(x_j - \frac{\sum_{i=2}^{m} w_i x_i}{\sum_{i=2}^{m} w_i}\right)^2\right)^2},$$

$$= \frac{\sigma^2}{2}\frac{1}{\sum_{j=2}^{m} w_j\left(x_j - \frac{\sum_{i=2}^{m} w_i x_i}{\sum_{i=2}^{m} w_i}\right)^2}.$$

To simplify the derivations in the rest of the article, we assume $P_j = \frac{w_j}{\sum_{j=2}^m w_j}$. Accordingly, we may rewrite $\hat{\beta}_{0w}$ as

$$\hat{\beta}_{0w} = \sum_{j=2}^m P_j Y_j^* - \hat{\beta}_{1w} \sum_{j=2}^m P_j x_j = \sum_{j=2}^m \left(P_j - \frac{\left(\sum_{i=2}^m P_i x_i\right) w_j \left(x_j - \frac{\sum_{i=2}^m w_i x_i}{\sum_{j=2}^m w_i} \right)}{\sum_{k=2}^m w_k \left(x_k - \frac{\sum_{i=2}^m w_i x_i}{\sum_{i=2}^m w_i} \right)^2} \right) Y_j^*$$

So

$$Var(\hat{\beta}_{0w}) = \frac{\sigma^2}{2} \sum_{j=2}^m D_j \left(P_j - \frac{\left(\sum_{i=2}^m P_i x_i\right) w_j \left(x_j - \frac{\sum_{i=2}^m w_i x_i}{\sum_{j=2}^m w_i} \right)}{\sum_{k=2}^m w_k \left(x_k - \frac{\sum_{i=2}^m w_i x_i}{\sum_{i=2}^m w_i} \right)^2} \right)^2$$

Applying straightforward simplifications allows us to see that

$$Var(\hat{\beta}_{0w}) = \frac{\sigma^2}{2} \left(\frac{1}{\sum_{j=2}^m w_j} - 2\left(\sum_{j=2}^m P_i x_i \right) \frac{\sum_{j=2}^m P_j \left(x_j - \frac{\sum_{i=2}^m w_i x_i}{\sum_{j=2}^m w_i} \right)}{\sum_{k=2}^m w_k \left(x_k - \frac{\sum_{i=2}^m w_i x_i}{\sum_{i=2}^m w_i} \right)^2} \right.$$

$$\left. + \left(\sum_{j=2}^m P_i x_i \right)^2 \frac{\sum_{j=2}^m w_j \left(x_j - \frac{\sum_{i=2}^m w_i x_i}{\sum_{j=2}^m w_i} \right)^2}{\left(\sum_{k=2}^m w_k \left(x_k - \frac{\sum_{i=2}^m w_i x_i}{\sum_{i=2}^m w_i} \right)^2 \right)^2} \right)$$

Eventually, we may conclude that

$$Var(\hat{\beta}_{0w}) = \frac{\sigma^2}{2} \left(\frac{1}{\sum_{j=2}^m w_j} + \frac{\left(\sum_{j=2}^m P_i x_i \right)^2}{\sum_{k=2}^m w_k \left(x_k - \frac{\sum_{i=2}^m w_i x_i}{\sum_{i=2}^m w_i} \right)^2} \right)$$

Theorem 2
Under the MERSS, we have

1) $Var(\hat{\beta}_{1w}) \leq Var(\hat{\beta}_1)$

2) $\mathit{eff}\ (\hat{\beta}_{1w},\ \hat{\beta}_1) = \frac{Var(\hat{\beta}_1)}{Var(\hat{\beta}_{1w})} \geq 1.$

where $\hat{\beta}_1 = \dfrac{\sum_{i=2}^{2}\sum_{j=2}^{m}(x_j - \bar{x})Y_{ij}}{2\sum_{j=2}^{m}(x_j - \bar{x})^2}$ is the least squares estimate of β_1 obtained using the SRS (x_j, Y_{ij}) for $i = 1, 2$ and $j = 2, 3, ..., m$ with $Var(\hat{\beta}_1) = \dfrac{\sigma^2}{2\sum_{j=2}^{m}(x_j - \bar{x})^2}$

PROOF:
1) It is sufficient to show that

$$\sum_{j=2}^{m} w_j\left(x_j - \frac{\sum_{i=2}^{m} w_i x_i}{\sum_{i=2}^{m} w_i}\right)^2 \geq \sum_{j=2}^{m}(x_j - \bar{x})^2.$$

Since $D_j \leq 1$ for all j, then $w_j \geq 1$ for all j.
So

$$\sum_{j=2}^{m} w_j\left(x_j - \frac{\sum_{i=2}^{m} w_i x_i}{\sum_{i=2}^{m} w_i}\right)^2 \geq \sum_{j=2}^{m}\left(x_j - \frac{\sum_{i=2}^{m} w_i x_i}{\sum_{i=2}^{m} w_i}\right)^2.$$

It is easy to see that the value of α which minimizes $\sum_{j=2}^{m}(x_j - \alpha)^2$ is $\bar{x}$, and this completes the proof.

2) The relative efficiency of $\hat{\beta}_{1w}$ with respect to $\hat{\beta}_1$ is simply concluded as a byproduct of the previous result of this theorem. In other words,

$$eff\ (\hat{\beta}_{1w},\ \hat{\beta}_1) = \frac{Var(\hat{\beta}_1)}{Var(\hat{\beta}_{1w})} \geq 1.$$

THEOREM 3
Assuming that $\hat{\beta}_0$ is the SRS estimator of $\hat{\beta}_0$ and using the MERSS, we have

1) $Var(\hat{\beta}_{0w}) \leq Var(\hat{\beta}_0)$,

2) $eff\ (\hat{\beta}_{0w},\ \hat{\beta}_0) = \dfrac{Var(\hat{\beta}_0)}{Var(\hat{\beta}_{0w})} \geq 1.$

PROOF:
We rewrite $Var(\hat{\beta}_0)$ and $Var(\hat{\beta}_{0w})$ as follows

$$Var(\hat{\beta}_{0w}) = \frac{\sigma^2}{2}\left(\frac{1}{(m-1)} + \frac{\bar{x}^2}{\sum_{j=2}^{m}(x_j - \bar{x})^2}\right)$$

$$= \frac{\sigma^2}{2(m-1)}\left(\frac{\sum_{j=2}^{m} x_j^2}{\sum_{j=2}^{m}(x_j - \bar{x})^2}\right)$$

and

$$Var(\hat{\beta}_{0w}) = \frac{\sigma^2}{2}\left(\frac{1}{\sum_{j=2}^{m} w_j} + \frac{\left(\sum_{j=2}^{m} P_j x_j\right)^2}{\sum_{j=2}^{m} w_j\left(x_j - \sum_{i=2}^{m} P_i x_i\right)^2}\right)$$
$$= \frac{\sigma^2}{2}\frac{\sum_{j=2}^{m} P_j x_j^2}{\sum_{j=2}^{m} w_j\left(x_j - \sum_{i=2}^{m} P_i x_i\right)^2}.$$

Since $P_j \leq 1$ for all j, then

$$\sum_{j=2}^{m} P_j x_j^2 \leq \sum_{j=2}^{m} x_j^2 \quad \text{and} \quad \sum_{j=2}^{m} w_j (x_j - \sum_{i=2}^{m} P_i x_i)^2 \geq \sum_{j=2}^{m} (x_j - \bar{x})^2.$$

Combining the last two inequalities shows that $Var(\hat{\beta}_{0w}) \leq Var(\hat{\beta}_0)$, which completes the proof.

Hence

$$eff\ (\hat{\beta}_{0w},\ \hat{\beta}_0) \geq 1.$$

4. ESTIMATION OF σ

In this section, we plan to develop a consistent estimator of the dispersion parameter σ. To this end, we propose the following estimator for the parameter σ

$$\hat{\sigma} = \frac{1}{2(m-1)} \sum_{j=2}^{m} \frac{(Y_{j:j} - Y_{1:j})}{A_j} \tag{4}$$

where $A_j = j \int_{-\infty}^{\infty} z \Phi^{j-1}(z) \varphi(z) dz$.

Theorem 4

1) The estimator $\hat{\sigma}$ given in (4) is an unbiased estimator of σ.

2) The variance of $\hat{\sigma}$ is $Var(\hat{\sigma}) = \dfrac{\sigma^2}{2(m-1)} \sum_{j=2}^{m} \dfrac{D_j}{A_j^2}$.

Proof:

1) It is easy to see that $\hat{\sigma}$ is an unbiased estimator of σ since

$$E(\hat{\sigma}) = \frac{1}{2(m-1)} \sum_{j=2}^{m} \frac{E(Y_{j:j} - Y_{1:j})}{A_j}$$
$$= \frac{1}{2(m-1)} \sum_{j=2}^{m} \frac{2\sigma A_j}{A_j}$$
$$= \sigma$$

2) To derive the variance of the estimator $\hat{\sigma}$, we use the fact that $Y_{j:j}$ and $Y_{1:j}$ are independent which implies that

$$Var(\hat{\sigma}) = \frac{1}{4(m-1)^2}\sum_{i=2}^{m}\frac{Var(Y_{j:j}) + Var(Y_{1:j})}{A_j^2}.$$

Moreover, we have pointed out in section 2 that $Var(\varepsilon_{j:j}) = Var(Y_{j:j}) = = Var(Y_{1:j}) = \sigma^2 D_j$ which allows us to conclude that

$$\begin{aligned} Var(\hat{\sigma}) &= \frac{1}{4(m-1)^2}\sum_{j=2}^{m}\frac{2\sigma^2 D_j}{A_j^2}, \\ &= \frac{\sigma^2}{2(m-1)^2}\sum_{j=2}^{m}\frac{D_j}{A_j^2}. \end{aligned}$$

and this concludes the theorem.

In another direction and to compare this estimator with its SRS counterpart, we consider a SRS $V_2, ..., V_{2m}$ from $N(\beta_0 + \beta_1 x,\ \sigma^2)$ where x is a fixed value of X. An estimator of σ based on $V_2, ..., V_{2m}$ is given by

$$\hat{\sigma}_{SRS} = K_m S$$

and $S = \frac{1}{2(m-1)}\sqrt{\sum_{j=2}^{2m}(V_j - \bar{V})^2}$ and $K_m = \frac{\Gamma\left(m - \frac{3}{2}\right)}{\sqrt{2}\Gamma(m-1)}$ Also, the variance of $\hat{\sigma}_{SRS}$ is given by

$$\begin{aligned} Var(\hat{\sigma}_{SRS}) &= \sigma^2\left[\left(m - \frac{3}{2}\right)\frac{\Gamma^2(m-2)}{\Gamma^2(m-1)} - 1\right] \\ &= \frac{2(m-1)^2\left(\left(m - \frac{3}{2}\right)\frac{\Gamma^2(m-2)}{\Gamma^2(m-1)} - 1\right)}{\sum_{j=2}^{m}\frac{D_j}{A_j^2}}. \end{aligned}$$

Table 1 shows the value of the $eff(\hat{\sigma}_{SRS}, \hat{\sigma})$ for different values of m. We see that the efficiency is always greater than 1, which means that the estimator $\hat{\sigma}$ is more efficient than $\hat{\sigma}_{SRS}$.

TABLE 1. - *The Efficiency of $\hat{\sigma}_{SRS}$, with respect to $\hat{\sigma}$.*

Set size: (m)	Efficiency: $eff(\hat{\sigma}_{SRS}, \hat{\sigma})$
2	1.6746
3	1.3893
4	1.5290
5	1.7261
6	1.9395
7	2.1583
8	2.3785
20	4.9262
50	10.6175

5. APPLICATION TO REAL DATA

As an illustration to the MERSS, we consider the collected data given by Samawi and Al-Sagheer (2001), where the relationship between the weight (X) and the level of bilirubin in the blood (Y) at birth of babies is of major interest to pediatricians. The bilirubin level in blood is determined according to a test which requires 30 minutes or more. In intensive case, this test should be taken several times for neonatal infants. In fact, measuring the variable Y is expensive, while ranking on Y is possible by the experience of physicians. Table 3 shows the values of X and Y for different infants. For $j = 2$, if we apply the MERSS scheme at $x = 2.0$ for the first group, then we get $y_{1:2} = 8.6$, $y_{2:2} = 15.76$. Similarly, from the second group and for $j = 3$ we have $x = 2.6$, $y_{1:3} = 10.94$, $y_{3:3} = 22.52$. Finally, from the third group where $j = 4$ and $x = 3.6$ we have $y_{1:4} = 16.2$, $y_{4:4} = 22.12$ Hence, the MERSS sample according to the scheme of interest is $y_1^* = 12.18$ $y_2^* = 16.73$ and $y_3^* = 19.16$. To compare the MERSS approach with the traditional sampling techniques, Table 2 give a set of size 6 pairs that represents our simple random sample

TABLE 2. - *Simple random sample of size 6 for the bilirubin level data set*

X	2.0	2.0	2.6	2.6	3.6	3.6
Y	8.6	15.76	16.76	14.12	16.50	19.46

To carry on the same direction, we fit simple linear regression model using the previous SRS and a straightforward computation gives the estimates $\hat{\beta}_0 = 5.59$ with an estimated standard error of 5.148 and $\hat{\beta}_1 = 3.51$ with an estimated standard error of 1.831. Also, we conducted the Anderson-Darling test of normality on the residuals and the test produced a ρ value of 0.867 meaning that we are not supposed to reject the normality assumption of the errors. Moreover, we used the Mathematica software to obtain the values $D_2 = 0.682$, $D_3 = 0.559$ and $D_4 = 0,442$ while the corresponding weights mentioned earlier in this article are $W_2 = 1.47$, $W_3 = 1.79$ and $W_4 = 2.03$. Accordingly, we obtain the weighted least squares estimates based on the previous results and we get $\hat{\beta}_{0w} = 5.11$ with an estimated standard error of 3.853 and $\hat{\beta}_{1w} = 4.00$ with an estimated standard error of 1.317. Eventually, we get the estimated efficiencies of β_0 and β_1 as mentioned earlier to be 1.337 and 1.390, respectively. The results here endorse the theoretical part of this paper.

TABLE 3. - *Weight (X) and bilirubin level in blood (Y) for different infants*

Group	Sample elements
1	(2,8.6), (2,11), (2,10.94), (2,15.76)
2	(2.6,15.47), (2.6,10.94), (2.6,12.12), (2.6,14.12), (2.6,16.76), (2.6, 22.52)
3	(3.6,16.2),(3.6, 19.46), (3.6, 16.46), (3.6, 16.5), (3.6, 18.29), (3.6, 22.12), (3.6, 21.29), (3.6, 18.35)

6. CONCLUSION

In this article, we considered a modified type of RSS, namely MERSS and implement this approach to the simple linear regression setup. We derived a new set of parameters estimators and showed that these estimators are more efficient than their counterparts under the SRS scheme. Also, more work on MERSS and regression analysis could be tractable in the future, especially in testing hypothesis and prediction.

ACKNOWLEDGEMENTS

We would like to thank the referee for his comments which helped us to improve the final structure and content of this article.

RIASSUNTO

Il campionamento moving extreme ranked set, introdotto da Alodat e Al-Saleh (2001), rappresenta una variante del ben noto campionamento ranked set proposto da McIntyre (1952). Nel presente lavoro, si propongono nuovi stimatori per i parametri del modello di regressione lineare semplice quando lo schema di campionamento utilizzato è il moving extreme ranked set. Inoltre, si dimostra che gli stimatori proposti sono più efficienti di quelli ottenuti utilizzando il campionamento casuale semplice. La proposta metodologica, completata con simulazioni e analisi dei dati, viene infine confrontata con gli approcci tradizionali.

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