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Kurtosis Diagram for the Log-Dagum Distribution

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Summary: *The aim of this paper is to study, through the kurtosis diagram proposed by Zenga (1996), the kurtosis of the log-Dagum distribution. In particular, we analyse how changes the kurtosis on the left and right side of the distribution when the parameter β changes, having fixed the median and absolute mean value from the median. We prove that the kurtosis depends only on parameter β and we show that a reduction of the kurtosis on the left side of the diagram and an increse of the kurtosis on the right side of the diagram occours when β increases.*

Keywords: *Log-Dagum distribution, kurtosis diagram, symmetric and asymmetric distribution.*

1. Introduction

In this paper we study the Log-Dagum distribution obtained by a logarithmic transformation of Dagum random variable (Dagum 1977, 1980). In particular, in *Section two* we will describe some characteristics of this distribution and, in *Section three*, we will determine the kurtosis diagram, proposed by Zenga (1996) following an approach based on the Pigou-Dalton principle of transfers.

Domma (2001) has shown that the Log-Dagum distribution can be symmetric, positive or negative skewed when the parameter β changes, the only parameter affecting the relative punctual asymmetries. We will see that the kurtosis depends only on parameter β too. Finally, we will highlight the importance of the study of the random variables S and D, defined as left and right deviation from median respectively, to obtain more information about the shape of the asymmetric density function.

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2. Characteristics of the Log-Dagum distribution

A positive random variable (*rv*) Y is Dagum distributed if its probability density function (*pdf*) is:

$$f_Y(y;\beta,\lambda,\theta)=\beta\lambda\theta y^{-\theta-1}\left(1+\lambda y^{-\theta}\right)^{-\beta-1}$$

with β>0, λ>0 and θ>0, where λ is a scale parameter and β and θ are shape parameters. The cumulative distribution function (*cdf*) is given by

$$F_Y\left(y;\beta,\lambda,\theta\right)=\left(1+\lambda y^{-\theta}\right)^{-\beta}.$$

A *rv* Y with Dagum distribution is indicated by Da(β,λ,θ).
The logarithmic transformation, X=ln(Y), has the following *cdf*

$$F_X(x;\beta,\lambda,\theta)=P_r\left\{X\le x\right\}=F_Y\left(e^{x};\beta,\lambda,\theta\right)=\left[1+\lambda e^{-\theta x}\right]^{-\beta} \qquad (1)$$

with *pdf*

$$f_X(x;\beta,\lambda,\theta)=\beta\lambda\theta e^{-\theta x}\left(1+\lambda e^{-\theta x}\right)^{-\beta-1} \qquad (2)$$

where $x\in\Re$ and β>0, λ>0 and θ>0. In this paper, the *rv* Log-Dagum will be denoted with LDa(β,λ,θ).

From (2) is simple to verify that the mode exists and is given by $m=\frac{1}{\theta}\ln(\lambda\beta)$. Moreover, solving respect to x the equation $F_X(x;\beta,\lambda,\theta)=q$, with $q\in(0,1)$, we obtain the q-th quantile, that is

$$x_{(q)}=\frac{1}{\theta}\ln(\lambda)-\frac{1}{\theta}\ln\left[q^{-\frac{1}{\beta}}-1\right]=\frac{1}{\theta}\ln\left(\frac{\lambda}{q^{-\frac{1}{\beta}}-1}\right),$$

putting q=0.5, the median is given by

$$\gamma=\frac{1}{\theta}\ln\left(\frac{\lambda}{2^{\frac{1}{\beta}}-1}\right). \qquad (3)$$

The moment generating function of a *rv* LDa(β,λ,θ) is equal to the moment of order t of the rv Da(β,λ,θ). In fact, we have

$$m_X(t) = E\left[e^{tX}\right] = E\left[e^{t\ln(Y)}\right] = E\left(Y^t\right) = \beta\lambda^{\frac{t}{\theta}}B\left(\beta + \tfrac{t}{\theta}, 1 - \tfrac{t}{\theta}\right)$$

for θ>t , where B(.,.) is Beta mathematical function. In this contest, to calculate the moments of a *rv* LDa(β,λ,θ) it is more convenient to use the cumulant generating function, $\ln[m_X(t)]$, from which the r-th cumulant is given by

$$K_r(X) = \left\{ \frac{\partial^r \ln\left[m_X(t)\right]}{\partial t^r} \right\}_{t=0}$$

[see e.g. Kendall and Stuart (1969), Chapter 3]. So, for instance, the first two cumulant, that is the first moment and the variance, are given by

$$K_1(X) = \left\{ \frac{\partial}{\partial t}\left[\ln(\beta) + \frac{t}{\theta}\ln(\lambda) + \ln\Gamma\left(\beta + \frac{t}{\theta}\right) + \ln\Gamma\left(1 - \frac{t}{\theta}\right) - \ln\Gamma\left(\beta+1\right) \right] \right\}_{t=0} =$$

$$= \left\{ \left[\frac{1}{\theta}\ln(\lambda) + \psi\left(\beta + \frac{t}{\theta}\right)\frac{1}{\theta} + \Psi\left(1 - \frac{t}{\theta}\right)\left(-\frac{1}{\theta}\right) \right] \right\}_{t=0} = \frac{\ln(\lambda)}{\theta} + \frac{1}{\theta}\left[\Psi(\beta) - \Psi(1)\right]$$

$$K_2(X) = \left\{ \frac{\partial}{\partial t}\left[\frac{\ln(\lambda)}{\theta} + \frac{1}{\theta}\psi\left(\beta + \frac{t}{\theta}\right) - \frac{1}{\theta}\Psi\left(1 - \frac{t}{\theta}\right) \right] \right\}_{t=0} = \frac{1}{\theta^2}\left[\Psi'(\beta) + \Psi'(1)\right]$$

where $\Psi(.)$ and $\Psi'(.)$ are, respectively, the digamma and trigamma function [see e.g. Davis (1970), pag. 258-274].

We observe, moreover, that the relative punctual asymmetries

$$W_X(p;\beta) = \frac{2\ln\left(2^{\frac{1}{\beta}} - 1\right) - \ln\left(p^{-\frac{1}{\beta}} - 1\right) - \ln\left[(1-p)^{-\frac{1}{\beta}} - 1\right]}{\ln\left(p^{-\frac{1}{\beta}} - 1\right) - \ln\left[(1-p)^{-\frac{1}{\beta}} - 1\right]} \qquad (4)$$

with $p \in \left(0, \frac{1}{2}\right)$, for β=1 are all equal to zero and, therefore, the *rv* log-Dagum is symmetric around $E(X) = m = \gamma = \frac{\ln(\lambda)}{\theta}$. From (2.4), we observe that β is the only parameter that influences the asymmetry of the *rv* LDa(β,λ,θ). Domma (2001) has shown that parameter β is a direct indicator of asymmetry; particularly, the LDa(β,λ,θ) model for $\beta \in (0,1)$ is negative skewed and for β>1 is positive skewed. This feature emphasizes the Log-Dagum model flexibility. In fact, varying parameter β, this model can describe situations of negative skewed, positive skewed and simmetries. F*igures 1a*, *1b* and *1c* shows the density function of the *rv* LDa(β,λ,θ), for different values of parameters β, λ and θ.

From (1) special distribution can be obtained, such as:

i) for $\beta = \lambda = 1$ and $\theta = \frac{\pi}{\sqrt{3}}$, we get the Logistic standard

$$f(y) = \frac{\pi}{\sqrt{3}} \frac{e^{-\frac{\pi}{\sqrt{3}}y}}{\left(1 + e^{-\frac{\pi}{\sqrt{3}}y}\right)^2};$$

ii) for $\theta = \frac{1}{\sigma}$ and $\lambda = e^{\frac{\mu}{\sigma}}$, we get the type I generalized logistic distribution

$$f(y) = \frac{\beta}{\sigma} \frac{e^{-\frac{y-\mu}{\sigma}}}{\left(1 + e^{-\frac{y-\mu}{\sigma}}\right)^{\beta+1}}.$$

Now, let r be a real constant, then incomplete first moment for the rv LDa(β,λ,θ) is

$$E_r(X) = \int_{-\infty}^{r} x f(x;\beta,\lambda,\theta) dx = \beta\lambda\theta \int_{-\infty}^{r} x \left(e^x\right)^{-\theta} \left[1 + \lambda\left(e^x\right)^{-\theta}\right]^{-\beta-1} dx$$

and, assuming w=exp(x), we get

$$E_r(X) = \beta\lambda\theta \int_{0}^{e^r} \ln(w) w^{-\theta-1} \left[1 + \lambda w^{-\theta}\right]^{-\beta-1} dw.$$

Moreover, putting $z=\left(1+\lambda w^{-\theta}\right)^{-1}$, with $dw=\theta^{-1}\lambda^{\frac{1}{\theta}}z^{\frac{1}{\theta}-1}(1-z)^{-\frac{1}{\theta}-1}dz$ and $z^*=\dfrac{e^{r\theta}}{\lambda+e^{r\theta}}<1$, after simple algebra, we have

$$E_r(X)=\frac{\beta}{\theta}\int_0^{z^*}\left[\ln(\lambda)+\ln(z)-\ln(1-z)\right]z^{\beta-1}dz=$$

$$=\frac{(z^*)^\beta\ln(\lambda)}{\theta}+\frac{\beta}{\theta}\left\{\int_0^{z^*}z^{\beta-1}\ln(z)dz-\int_0^{z^*}z^{\beta-1}\ln(1-z)dz\right\}=$$

$$=\frac{(z^*)^\beta\ln(\lambda)}{\theta}+\frac{1}{\theta}\left\{(z^*)^\beta\left[\ln\left(\frac{z^*}{1-z^*}\right)-\frac{1}{\beta}\right]-\sum_{j=0}^{\infty}\frac{(z^*)^{\beta+j+1}}{(\beta+j+1)}\right\}. \qquad (5)$$

This last expression is obtained through integration by parts and considering that the geometric series $\frac{1}{1-z}=\sum_{j=0}^{\infty}z^j$ is uniformly convergent for $z\in(0,z^*)$, with z*<1. We observe, moreover, that the series $\sum_{j=0}^{\infty}\frac{(z^*)^{\beta+j+1}}{(\beta+j+1)}$ is convergent because it is a power series with radius of convergence z*<1.

Putting r=γ in (5), we obtain

$$E_\gamma(X)=\frac{\ln(\lambda)}{2\theta}+\frac{1}{\theta}\left\{\frac{1}{2}\left[\ln\left(\frac{1}{2^{\frac{1}{\beta}}-1}\right)-\frac{1}{\beta}\right]-\sum_{j=0}^{\infty}\frac{\left(2^{-\frac{1}{\beta}}\right)^{\beta+j+1}}{(\beta+j+1)}\right\} \qquad (6)$$

because $z^*=2^{-\frac{1}{\beta}}$.

3. The Kurtosis diagram

Zenga (1996) has observed that, for the normal distribution, an increase or a reduction of the kurtosis can be see as a transfer of density that involves both "right-side" and "left-side" of the mean and such that preserves both mean and standard deviation. In particular, an increase of kurtosis on the right side of density, occurs if part of the area of density, correspondent to the intermediate values, shifts towards both the mean and the more extreme values. On the other side, a decrese of kurtosis occurs if part of the area of density, correspondent to the near values of the mean and to the more extreme values, shifts towards the intermediate values. With the aim of generalizing such observations, Zenga (1996) introduces specific trasformations that increase (decrease) the kurtosis, such trasformations are applied to the variables D and S; moreover, using the analogy with the "principle of transfers", he costructs the so-called diagram of kurtosis based on the Lorenz curves of the *rv* D and S.

We highlight that this approach may be utilized both for symmetric and asymmetric distributions.

Let X be a rv, not necessarily symmetric, with *cdf* $F(x;\boldsymbol{\xi})$ and *pdf* $f(x;\boldsymbol{\xi})$, with $\boldsymbol{\xi}\in\Xi\subset\Re^r$ and median γ.

The expectation value of random variable (X-γ) for X>γ is

$$\delta_1 = 2\int_\gamma^\infty (x-\gamma)f(x;\boldsymbol{\xi})dx = 2\int_\gamma^\infty xf(x;\boldsymbol{\xi})dx - 2\gamma\int_\gamma^\infty f(x;\boldsymbol{\xi})dx =$$

$$= 2\left\{\int_{-\infty}^\infty xf(x;\boldsymbol{\xi})dx - \int_{-\infty}^\gamma xf(x;\boldsymbol{\xi})dx\right\} - 2\gamma[1-F(\gamma;\boldsymbol{\xi})] =$$

$$= 2\{E(X) - E_\gamma(X)\} - \gamma \ . \tag{7}$$

Moreover, the expectation of (γ-X) for $X\le\gamma$ is given by

$$\delta_2 = 2\int_{-\infty}^\gamma (\gamma-x)f(x;\boldsymbol{\xi})dx = 2\gamma F(\gamma;\boldsymbol{\xi}) - 2\int_{-\infty}^\gamma xf(x;\boldsymbol{\xi})dx =$$

$$= \gamma - 2E_\gamma(X) \ . \tag{8}$$

From (7) and (8), we observe that the absolute mean deviation from the median, $\delta = \int_{-\infty}^{+\infty}|x-\gamma|f(x;\xi)dx$, may be written as

$$\delta = \frac{\delta_1+\delta_2}{2} = E(X) - 2E_\gamma(X) . \tag{9}$$

Finally, from (8), we stress that it is possible to get $2E_{\gamma}(X)=\gamma-\delta_2$ and, therefore, $2[E(X)-\gamma]=\delta_1-\delta_2$.
In order to analyze the behavior of density on left and right of the median, Zenga (1996) introduces two conditioned non-negative random variables, defined as deviation right and left by median, respectively, given by: D=(X-γ)/X>γ and $S=(\gamma-X)/X\leq\gamma$. The *cdf* and *pdf* of the *rv* D, respectively, are:

$$F_1(d;\xi)=\begin{cases}2F(d+\gamma;\xi)\text{ -1} & \text{for } d\geq 0\\ 0 & \text{otherwise}\end{cases}$$

$$f_1(d;\xi)=2f(d+\gamma;\xi)\,.$$

For the rv S, we have

$$F_2(s;\xi)=\begin{cases}1-2F(\gamma-s;\xi) & \text{for } s\geq 0\\ 0 & \text{otherwise}\end{cases}$$

$$f_2(d;\xi)=2f(\gamma-s;\xi)$$

[Polisicchio and Zenga (1997)]. Clearly, the expected values of *rv* D and S are δ_1 and δ_2, respectively. To obtain the kurtosis diagram, we consider the ordinates of the Lorenz curve for the non-negative *rv* D

$$L_D(p_1;\xi)=\frac{1}{\delta_1}\int_0^{p_1}F_1^{-1}(t;\xi)dt \qquad p_1\in[0,1] \tag{10}$$

where $F_1^{-1}(t;\xi)$ is the t-th quantile with $t\in[0,1]$ of the *rv* D. Likewise, denoted with $F_2^{-1}(t;\xi)$ the inverse of $F_2(s,\xi)$, the ordinates of the Lorenz curve of the *rv* S are given by

$$L_S(p_2;\xi)=\frac{1}{\delta_2}\int_0^{p_2}F_2^{-1}(t;\xi)dt \qquad p_2\in[0,1] \tag{11}$$

Following Zenga (1996), we consider a cartesian plane, on the axis of the abscissas, we take two half-lines having the same origin and opposite direction. On the points at “right” and “left” of the origin, we put the values of p_1 and p_2, respectively. On the axis of the ordinates, we transfer the

Lorenz curves (10) and (11), respectively. In this way, we get the kurtosis diagram for the continuous random variables.

4. The Kurtosis diagram for the random variable LDa(β,λ,θ)

Replacing in (7) the relations E(X), $E_\gamma(X)$ and γ, obtained in *Section two*, after simple algebra, we obtain

$$\delta_1 = \frac{1}{\theta}\left\{ 2[\Psi(\beta) - \psi(1)] + 2\ln\left(2^{\frac{1}{\beta}} - 1\right) + \frac{1}{\beta} + 2\sum_{j=0}^{\infty} \frac{\left(2^{-\frac{1}{\beta}}\right)^{\beta+j+1}}{(\beta+j+1)} \right\} \quad (12)$$

Likewise, δ_2 is

$$\delta_2 = \frac{1}{\theta}\left\{ \frac{1}{\beta} + 2\sum_{j=0}^{\infty} \frac{\left(2^{-\frac{1}{\beta}}\right)^{\beta+j+1}}{(\beta+j+1)} \right\}. \quad (13)$$

The variable D=(X-γ)/X>γ for the LDa(β,λ,θ) model, has *cdf* and *pdf*, respectively, given by

$$F_1(d;\beta,\lambda,\theta) = 2\left[1 + \lambda e^{-\theta(\gamma+d)}\right]^{-\beta} - 1 \quad (14)$$

$$f_1(d;\beta,\lambda,\theta) = 2\beta\lambda\theta e^{-\theta(\gamma+d)}\left[1 + \lambda e^{-\theta(\gamma+d)}\right]^{-\beta-1}. \quad (15)$$

From (14) we can verify that the $p_1 -$th quantile, with $p_1 \in (0,1)$, is given by

$$d_{(p_1)} = F_1^{-1}(p_1;\beta,\lambda,\theta) = -\frac{1}{\theta}\ln\left[\left(\frac{1+p_1}{2}\right)^{-\frac{1}{\beta}} - 1\right] - \gamma + \frac{\ln(\lambda)}{\theta}. \quad (16)$$

Likewise, for the rv $S = (\gamma - X)/X \le \gamma$ we have

$$F_2(s;\beta,\lambda,\theta) = 1 - 2\left[1 + \lambda e^{-\theta(\gamma-s)}\right]^{-\beta} \quad (17)$$

$$f_2(s;\beta,\lambda,\theta) = 2\beta\lambda\theta e^{-\theta(\gamma-s)}\left[1+\lambda e^{-\theta(\gamma-s)}\right]^{-\beta-1}. \qquad (18)$$

In addition, the p_2-th quantile, with $p_2 \in (0,1)$, is

$$s_{(p_2)} = F_2^{-1}(p_2;\beta,\lambda,\theta) = \gamma + \frac{1}{\theta}\ln\left[\left(\frac{1-p_2}{2}\right)^{-\frac{1}{\beta}} - 1\right] - \frac{\ln(\lambda)}{\theta}.$$

In order to derive the Lorenz curves of rv S and D, we observe that for $p_1 \in [0,1]$,

$$I_1(p_1;\beta) = \int_0^{p_1} \ln\left[\left(\tfrac{1+t}{2}\right)^{-\frac{1}{\beta}} - 1\right]dt = 2\left\{\sum_{j=0}^{\infty}\frac{\left(\frac{1+p_1}{2}\right)^{\frac{1+j}{\beta}+1}}{(\beta+j+1)} - \sum_{j=0}^{\infty}\frac{\left(\frac{1}{2}\right)^{\frac{1+j}{\beta}+1}}{(\beta+j+1)}\right\} - \ln\left[1-\left(\tfrac{1}{2}\right)^{\frac{1}{\beta}}\right] +$$

$$+ (1+p_1)\ln\left[1-\left(\tfrac{1+p_1}{2}\right)^{\frac{1}{\beta}}\right] - \frac{1}{\beta}\left\{(1+p_1)\left[\ln\left(\tfrac{1+p_1}{2}\right)-1\right] + \ln(2)+1\right\} =$$

$$= 2\left\{\sum_{j=0}^{\infty}\frac{\left(\frac{1+p_1}{2}\right)^{\frac{1+j}{\beta}+1}}{(\beta+j+1)} - \sum_{j=0}^{\infty}\frac{\left(\frac{1}{2}\right)^{\frac{1+j}{\beta}+1}}{(\beta+j+1)}\right\} - \ln\left[2^{\frac{1}{\beta}} - 1\right] + (1+p_1)\ln\left[\left(\frac{1+p_1}{2}\right)^{-\frac{1}{\beta}} - 1\right] + \frac{p_1}{\beta}$$

and for $p_2 \in [0,1]$

$$I_2(p_2;\beta) = \int_0^{p_2} \ln\left[\left(\tfrac{1-t}{2}\right)^{-\frac{1}{\beta}} - 1\right]dt = 2\left\{\sum_{j=0}^{\infty}\frac{\left(\frac{1}{2}\right)^{\frac{1+j}{\beta}+1}}{(\beta+j+1)} - \sum_{j=0}^{\infty}\frac{\left(\frac{1-p_2}{2}\right)^{\frac{1+j}{\beta}+1}}{(\beta+j+1)}\right\} + \ln\left[2^{\frac{1}{\beta}} - 1\right] +$$

$$+ \frac{p_2}{\beta} - (1-p_2)\ln\left[\left(\tfrac{1-p_2}{2}\right)^{-\frac{1}{\beta}} - 1\right]$$

(for more details see Appendix).
Considering $I_1(p_1;\beta)$, the Lorenz curve of *rv* D is given by

$$L_D(p_1) = \frac{1}{\delta_1}\int_0^{p_1}\left\{-\tfrac{1}{\theta}\ln\left[\left(\tfrac{1+t}{2}\right)^{-\frac{1}{\beta}} - 1\right] - \gamma + \tfrac{\ln(\lambda)}{\theta}\right\}dt =$$

$$= \frac{1}{\theta\delta_1}\left\{- I_1(p_1;\beta) + p_1 \ln\left(2^{\frac{1}{\beta}} - 1\right)\right\}. \tag{19}$$

with $p_1 \in [0,1]$. Similarly, the Lorenz curve of the rv S, is

$$L_S(p_2) = \frac{1}{\delta_2}\int_0^{p_2}\left\{\frac{1}{\theta}\ln\left[\left(\frac{1-t}{2}\right)^{-\frac{1}{\beta}} - 1\right] + \gamma - \frac{1\ln(\lambda)}{\theta}\right\}dt =$$

$$= \frac{1}{\theta\delta_2}\left\{I_2(p_2;\beta) - p_2 \ln\left(2^{\frac{1}{\beta}} - 1\right)\right\}. \tag{20}$$

with $p_2 \in [0,1]$. Remembering the expressions for δ_1 and δ_2, we can conclude that (19) and (20) do not depend on λ and θ. Consequently, the kurtosis function depends only on parameter β.
The first relevant result is that in the LDa(β,λ,θ) model both skewness and kurtosis are influenced only by parameter β.

5. Behaviour of the diagram of kurtosis when the parameter β varies

From the adopted approach it results that the analysis of the behavior of the kurtosis, varying the parameters, must be led for those subset of parameters such that both median and absolute mean deviation from the median remain constant. With this aim, we have followed the riparametrization proposed by Pollastri (1998) which assures the identification of triples of parameters such that γ and δ remain fixed at the values γ^* and δ^*, respectively. Table 1 shows some triples for the parameters values (β,λ,θ) such that the median is $\gamma^*=0.30702000$ and the absolute mean deviation from the median is $\delta^*=0.54888000$.

As can be noticed from figures *2a*, *2b* and *2c*, which report the *pdf* of *rv* Log-Dagum for some values of the parameters, it is not easy to understand the shape of the distribution when β changes, even when the median and the absolute mean deviation from the median are kept fixed.

From kurtosis diagram (*fig. 3a*, *3b* and *3c*) we observe, moreover, a different behaviour on the right and left sides of the diagram when β varies. In particular, when β □increases□the curve moves upwards on the left side of kurtosis diagram and, therefore, it points out a reduce of the kurtosis, while on the right side can be obeserved an increase of kurtosis.

Table 1 *Triples for the parameter values (β, λ ,θ)*

β	λ	θ	γ*	δ*
0.4	15.90000350	3.99967170	0.30702000	0.54888000
0.6	5.71350520	3.14603305	0.30702000	0.54888000
0.8	3.20678910	2.75010240	0.30702000	0.54888000
1.0	2.17152700	2.52566810	0.30702000	0.54888000
1.2	1.62467755	2.38247995	0.30702000	0.54888000
1.4	1.29162520	2.28369800	0.30702000	0.54888000
1.6	1.06921423	2.21166177	0.30702000	0.54888000
…	……	……..	……	……..
4.0	0.34223174	1.93031682	0.30702000	0.54888000
…	……	……..	……	……..
15	0.082350366	1.80638684	0.30702000	0.54888000

In such situation, in order to understand the shape of distribution, Pollastri (1998) has proposed to evaluate the kurtosis to varying of a parameter, fixed γ and δ, considering separately the *rv* S and D. So, with regard to the variable S, we put:
$S=S_1$ when $\beta=\beta^*$, $\lambda=\lambda^*$ and $\theta=\theta^*$, triple of parameters such that $\gamma=\gamma^*$ and $\delta=\delta^*$;
$S=S_2$ when $\beta=\overline{\beta}$, $\lambda=\overline{\lambda}$ and $\theta=\overline{\theta}$, triple of parameters such that $\gamma=\gamma^*$ and $\delta=\delta^*$.
Since, for skewed distributions, S_1 and S_2 have different expected values, that is $E[S_1]={}_{(S_1)}\delta_2 \neq E[S_2]={}_{(S_2)}\delta_2$, the kurtosis is not comparable directly when observing the density of S_1 and S_2. The transformation

$$S_1^{'}=aS_1=\frac{{}_{(S_2)}\delta_2}{{}_{(S_1)}\delta_2}S_1$$

shows expected values equal to ${}_{(S_2)}\delta_2$ and, therefore, we can compare the kurtosis of densities $f_2\left(s_1^{'};\beta^*,\lambda^*,\theta^*\right)$ for $S_1^{'}$ and $f_2\left(s_2;\overline{\beta},\overline{\lambda},\overline{\theta}\right)$ for S_2 when the parameter β varies, fixed γ and δ.
Analogous arguments can be used for studying the kurtosis of the rv D.

Figures *4a*, *4b* and *4c*, relative to variable S emphasize a reduction of the kurtosis on the left side of the distribution, when parameter β □increases. In fact, it is evident the transfer of density from extreme values towards the center of the distribution as parameter β increases. Relatively to the variable D, figures *5a*, *5b* and *5c* emphasize, how increasing parameter β, a transfer of density from the center of distribution towards the extreme values occurs and, therefore, point out an increase of kurtosis.

6. Conclusions

In the first part of the paper, we have analysed some characteristics of the Log-Dagum model, stressing its great flexibility. In the second part, we have studied the kurtosis through the kurtosis diagram proposed by Zenga (1996). In particular, we have shown that: (*i*) the kurtosis depends only on parameter β; (*ii*) when β increases we get a reduction of the kurtosis on the left side of the diagram, and an increase of the kurtosis on the right side of the diagram. This different behaviour has also been confirmed through a separate study of the variables S and D, respectively, defined device left and right of the median.

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Appendix A

In this section, we give details of the calculation of $I_1(p_1;\beta)$ and, for analogy, of $I_2(p_2;\beta)$ utilized in Section 4. Putting $z=\frac{1+t}{2}$ in $I_1(p_1;\beta)$, we have

$$I_1(p_1;\beta)=\int_0^{p_1}\ln\left[\left(\tfrac{1+t}{2}\right)^{-\frac{1}{\beta}}-1\right]dt=2\int_{1/2}^{\frac{1+p_1}{2}}\ln\left(z^{-\frac{1}{\beta}}-1\right)dz=$$

$$=2\int_{1/2}^{\frac{1+p_1}{2}}\ln\left(1-z^{\frac{1}{\beta}}\right)dz-\frac{2}{\beta}\int_{1/2}^{\frac{1+p_1}{2}}\ln(z)dz\,. \qquad (21)$$

the second integral in (21) is simple. For the first, we get $w=z^{\frac{1}{\beta}}$ and we integrate by parts, that is

$$\int_{1/2}^{\frac{1+p_1}{2}} \ln\left(1 - z^{\frac{1}{\beta}}\right) dz = \beta \int_{\left(\frac{1}{2}\right)^{\frac{1}{\beta}}}^{\left(\frac{1+p_1}{2}\right)^{\frac{1}{\beta}}} w^{\beta-1} \ln(1-w) dw =$$

$$= \beta \left\{ \frac{w^\beta}{\beta} \ln(1-w) \Bigg|_{\left(\frac{1}{2}\right)^{\frac{1}{\beta}}}^{\left(\frac{1+p_1}{2}\right)^{\frac{1}{\beta}}} + \frac{1}{\beta} \int_{\left(\frac{1}{2}\right)^{\frac{1}{\beta}}}^{\left(\frac{1+p_1}{2}\right)^{\frac{1}{\beta}}} \frac{w^\beta}{1-w} dw \right\} =$$

$$= w^\beta \ln(1-w) \Bigg|_{\left(\frac{1}{2}\right)^{\frac{1}{\beta}}}^{\left(\frac{1+p_1}{2}\right)^{\frac{1}{\beta}}} + \sum_{j=0}^{\infty} \frac{w^{\beta+j+1}}{(\beta+j+1)} \Bigg|_{\left(\frac{1}{2}\right)^{\frac{1}{\beta}}}^{\left(\frac{1+p_1}{2}\right)^{\frac{1}{\beta}}}$$

after simple algebra, we obtain the relation used in *Section 4*.

Appendix B

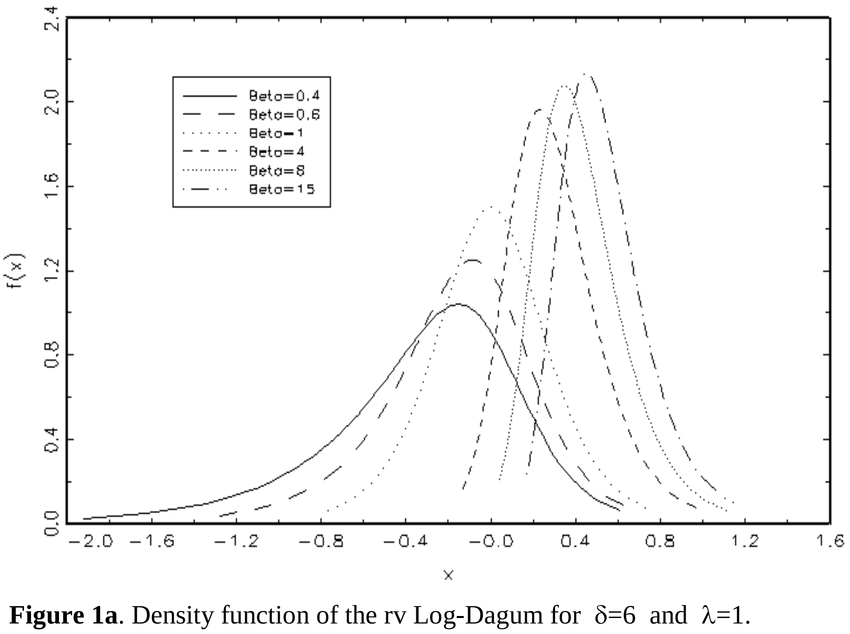

Figure 1a. Density function of the rv Log-Dagum for δ=6 and λ=1.

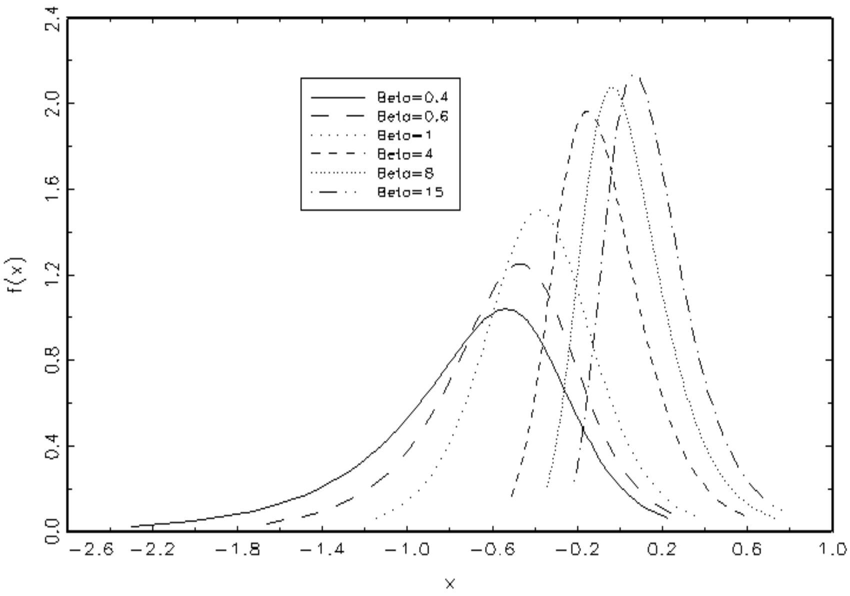

Figure 1b. Density function of the rv Log-Dagum for δ=6 and λ=0.1.

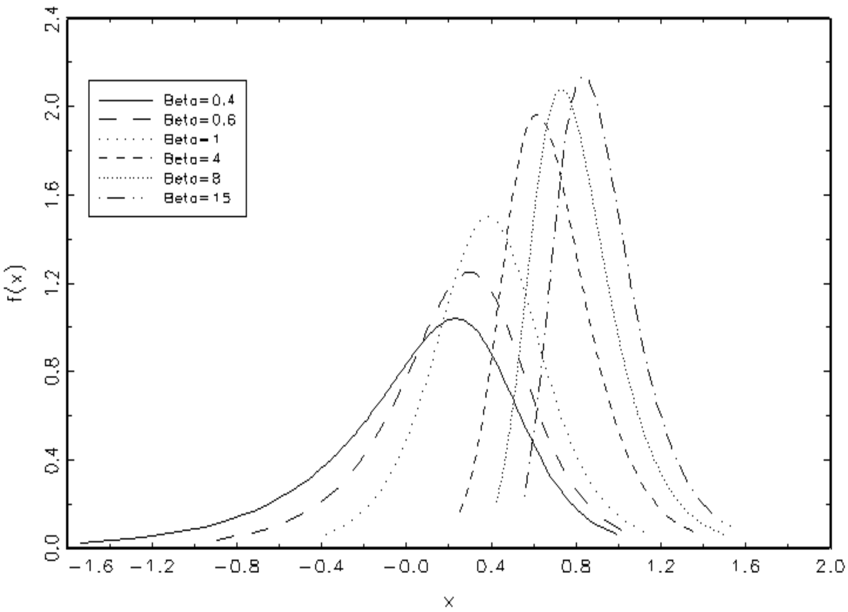

Figure 1c. Density function of the rv Log-Dagum for δ=6 and λ=10.

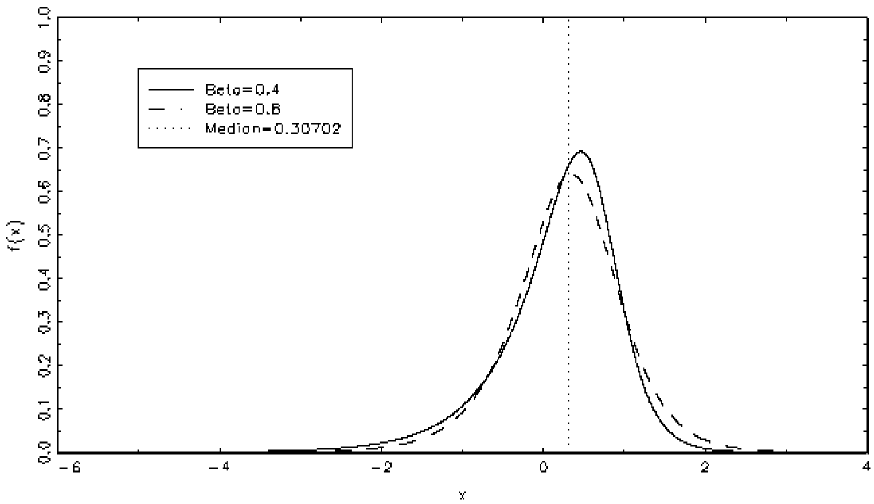

Figure 2a. Density function of the *rv* Log-Dagum with median=0.30702 and absolute mean deviation from the median=0.54888.

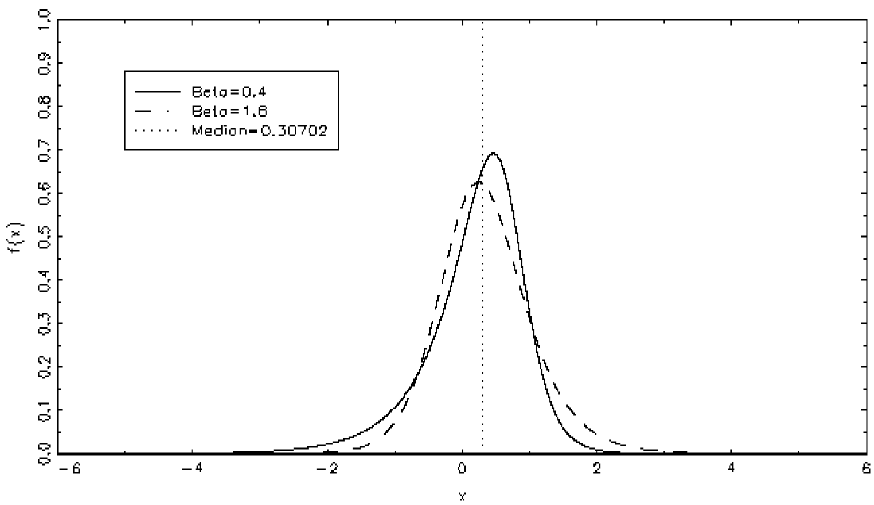

Figure 2b. Density function of the *rv* Log-Dagum with median=0.30702 and absolute mean deviation from the median=0.54888.

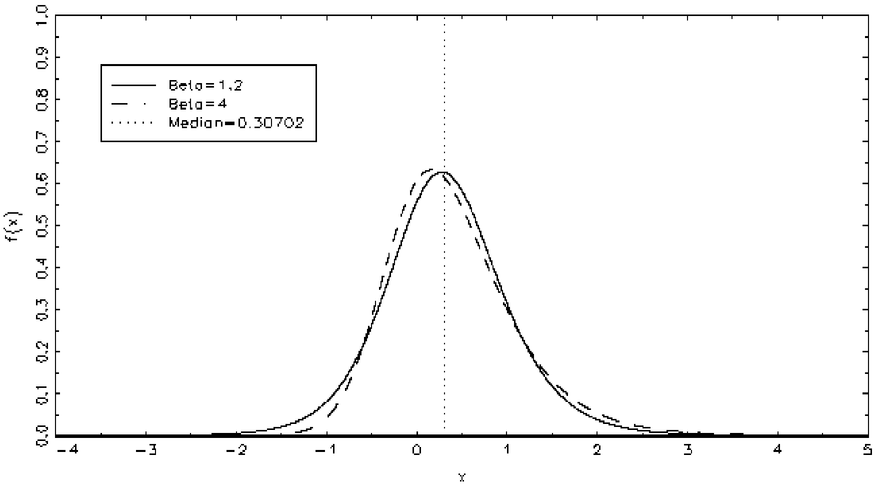

Figure 2c. Density function of the *rv* Log-Dagum with median=0.30702 and absolute mean deviation from the median=0.54888.

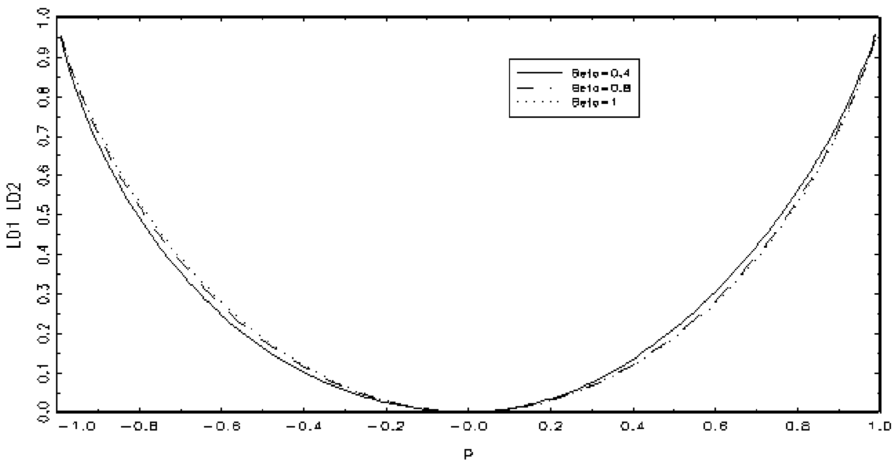

Figure 3a. Kurtosis diagram for the *rv* Log-Dagum

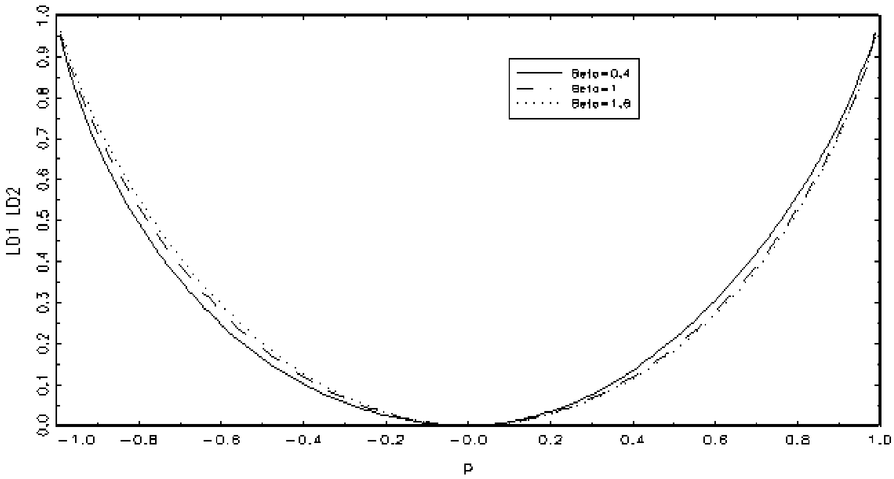

Figure 3b. Kurtosis diagram for the *rv* Log-Dagum

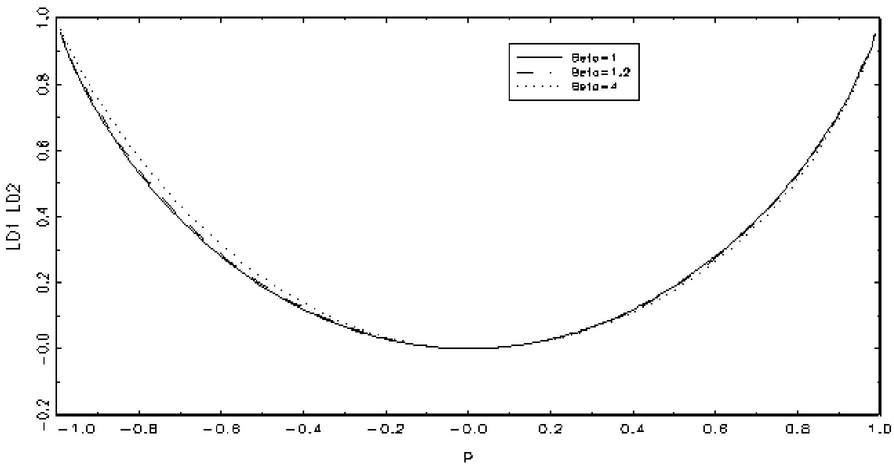

Figure 3c. Kurtosis diagram for the *rv* Log-Dagum

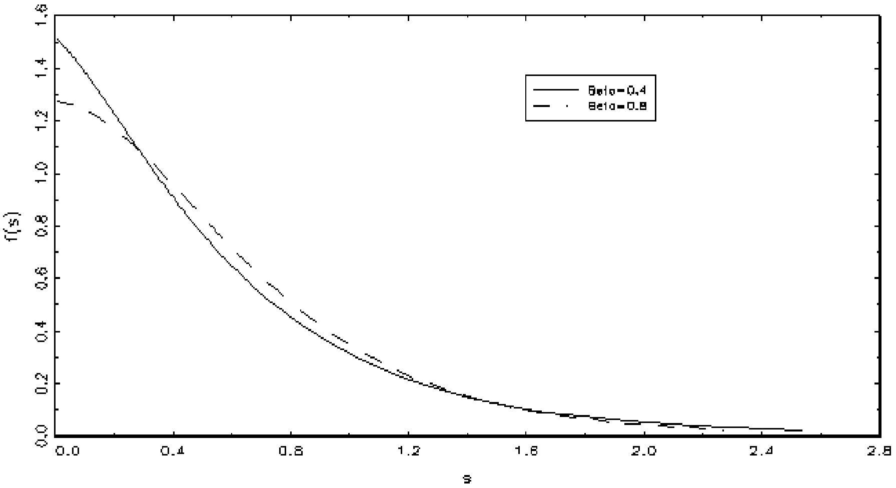

Figure 4a. Density function of *rv* S and S'.

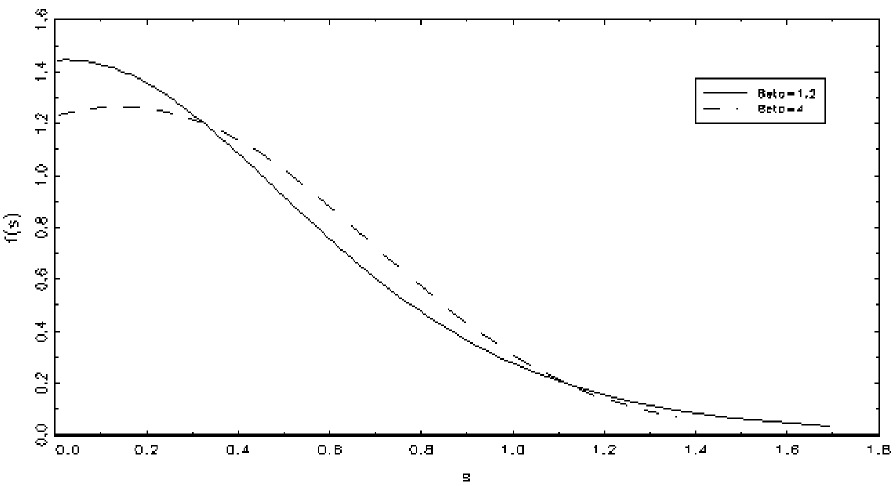

Figure 4b. Density function of *rv* S and S'.

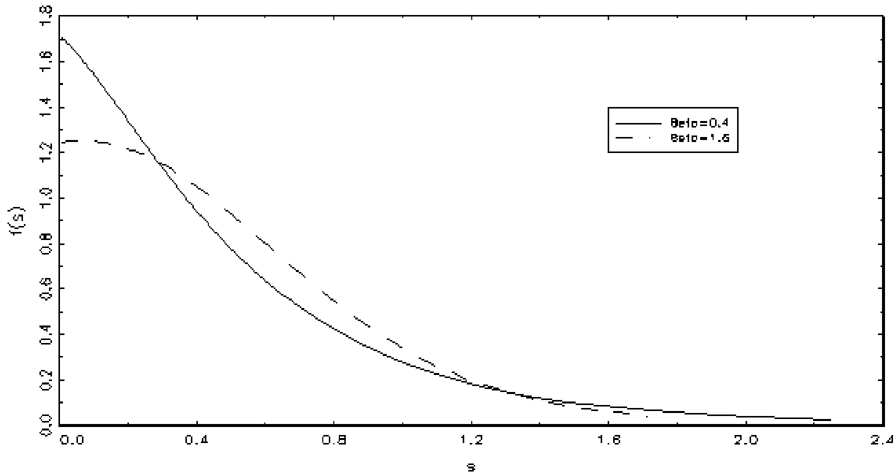

Figure 4c. Density function of *rv* S and S’.

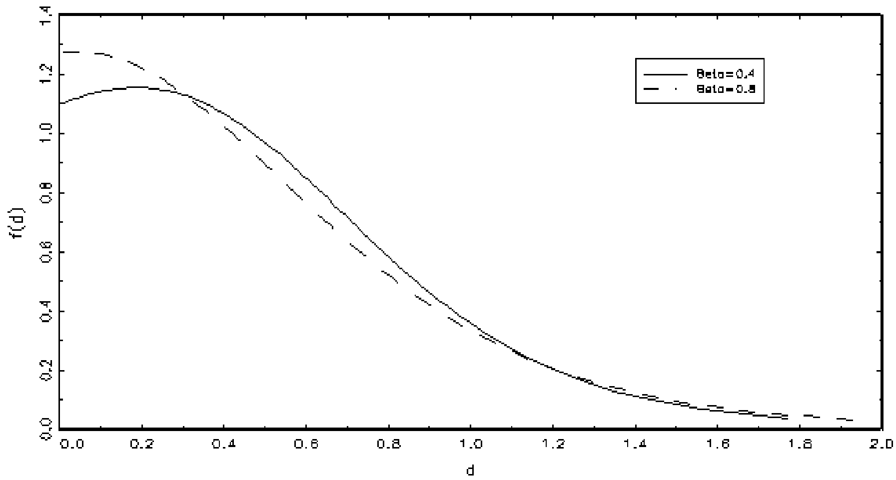

Figure 5a. Density function of *rv* D and D’.

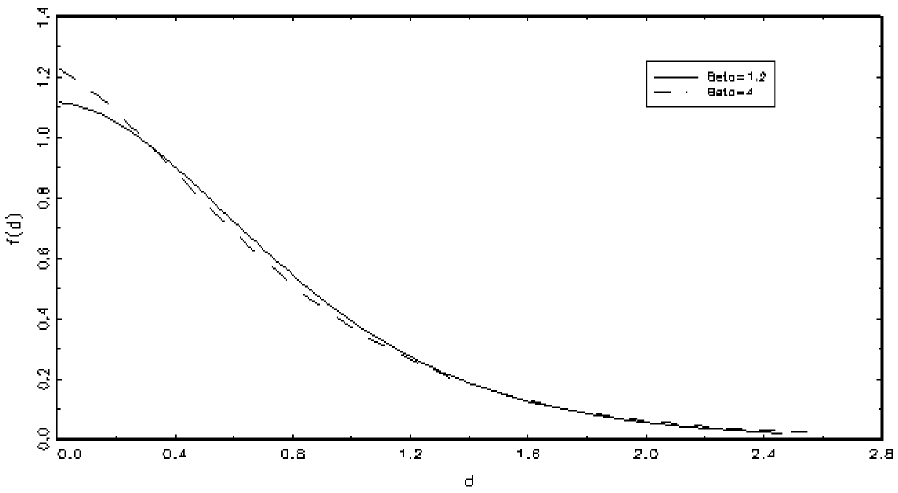

Figure 5b. Density function of *rv* D and D’.

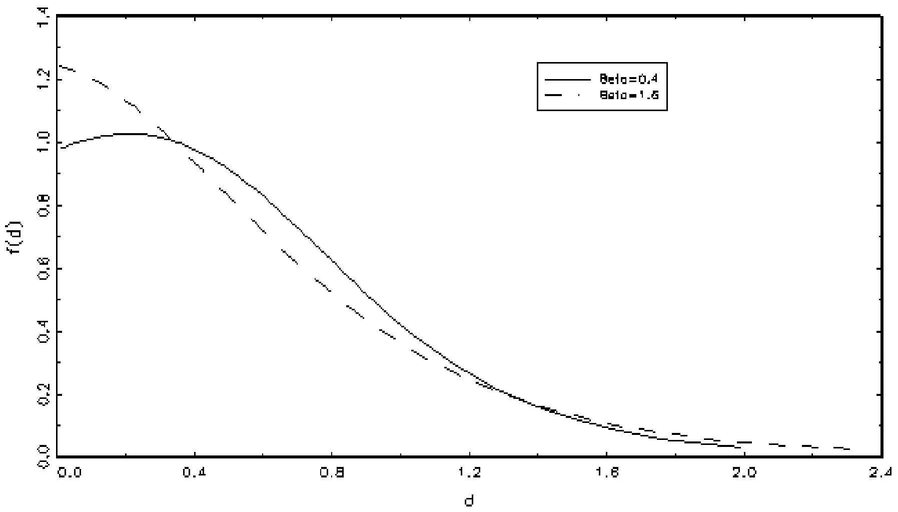

Figure 5c. Density function of *rv* D and D’.

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