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BIPOLAR DISTRIBUTIONS IN FUZZY SETS THEORY

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SUMMARY

In this paper we consider the problem of measuring latent variables with ordinal scales and we put forward an original approach to this issue, which combines the use of Intuitionistic Fuzzy Sets with the calculation of bipolar means and bipolar distributions. Intuitionistic Fuzzy theory allows a researcher to model the degree of membership and non-membership to a certain fuzzy set, as well as the residual uncertainty. It is fundamental, for decision making, to properly model such source of variability. We focus on the definition of uncertainty, using bipolar distributions and introducing Intuitionistic Bipolar Fuzzy Sets. This allows us to distinguish between a negative and a positive component of uncertainty, which represents a novelty in Intuitionistic Fuzzy analysis. We apply this method to a national evaluation survey (The Magellano Project) proposed by the Italian Ministry of Public Administration, aimed at involving employees in management decision.

Keywords: *Bipolar Means, Bipolar Distributions, Fuzzy Analysis, Intuitionistic Fuzzy Sets, Questionnaire Analysis, Evaluation, Decision Making, Management.*

1. INTRODUCTION

Quality evaluation and improvement is a fundamental topic in applied statistics with relevant implications in both the management of Public Services and in the private/industrial sector (Kenett, Zacks and Amberti, 2014; Kenett and Salini, 2013; Montgomery, 2013). In this paper we focus on the evaluation of Public Administrations and we propose an original and new statistical method, which combines the use of Intuitionistic Fuzzy Sets (IFS) with bipolar distributions.

Fuzzy Sets (FS) were introduced by Zadeh (1965) and Goguen (1967), and many applications have been proposed in several fields (e.g., Supciller and Nilsu, 2015; Zhang, Xia, Wang and Deng, 2015). Despite its name, fuzzy theory can be conceived as a formal and rigorous mathematical instrument aiming at representing phenomena defined by different degrees of membership to a certain set (Bede, 2013). In FS, the classical concept of membership, which is dichotomous, is indeed generalized to include different degrees of membership. For instance, in classical set theory, the integer 3, but not 3.2, belongs to the set $\mathbb{N}$ of natural numbers. In the fuzzy context there

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is not such a sharp distinction; for instance, a 30 years old man can belong to the set of young people with a certain degree, provided it is specified when a subject should be considered as ‘‘young’’.

Decision making is a good example of application of FS theory, because decisions are normally taken in conditions in which objectives, constraints and consequences of possible actions are not precisely known, but are sufficiently nuanced (fuzzy) (Bellman and Zadeh, 1970). As part of customer satisfaction research, Chien and Tsai (2000) studied the problem of measuring the perceived quality of a service by considering the level of satisfaction and the importance degree. Kwong and Bai (2002) used fuzzy techniques to determine the weights of customers' requirements in the context of the quality evaluation of a service. Baliamoune-Lutz (2006) used fuzzy analysis in well-being assessment.

Other authors employed FS to study poverty and income distribution (Cerioli and Zani, 1990; Lemmi and Betti, 2006), well-being (Chiappero-Martinetti, 2000, 2006), customer satisfaction (Zani, Milioli and Morlini, 2013), or to measure students' satisfaction with respect to the value of their University education in finding an employment (Crocetta and Delvecchio, 2007). Other important applications of fuzzy theory can be found in computer science, medicine, engineering control, expert systems, logic, management science and robotics.

Atanassov (1986) put forth a generalization of FS, by introducing IFS, which provide both a degree of membership and a degree of non-membership. In this framework, the sum of the values of membership and non-membership can be inferior to 1, and the residual component is defined as uncertainty. To clarify the concept of uncertainty, let's consider a putative employee, who is asked to quantify (between 0 and 1) the organization level of some aspects of the management in his/her own company. The answer could be 0.5 as a measure of satisfaction with regard to certain aspects of the organization, but 0.3 of dissatisfaction for others, leaving open a margin for uncertainty of 0.2. It is worth observing that the interpretation of uncertainty in fuzzy theory does not have a direct probability counterpart, but it is instead referred to the vagueness / indeterminacy of a decision.

In the last decade, a wide range of IFS applications in decision making have been developed. Indeed, in every decision process there does exist a certain degree of uncertainty. For instance, this theory has been applied in multi-attribute decision making (Xu, 2014) and in game theory management (Li, 2014). Other applications have been proposed in performance evaluation (Hajiagha, Akrami, Zavadskas and Hossein, 2013), in neural networks and expert systems (Atanassov, 2012).

In this paper, we focus on the definition of uncertainty in IFS, proposing an original quantitative interpretation of this concept by adopting bipolar means, introduced in the statistical literature by Maffenini and Zenga (2005, 2006) and Zenga, (2014), generalized by Zini (2008) and used in several contexts (e.g., Brentari, Dancelli and Dabrassi, 2006; Brentari and Dancelli, 2006; 2015). Section 2 is dedicated to introducing FS and IFS theory; Section 3 deals with bipolar means and bipolar distributions and presents our original idea of Intuitionistic Bipolar Fuzzy Sets (IBFS); in Section 4, bipolar distributions are introduced in the fuzzy field referring to questionnaires for satisfaction measurement. Section 5 proposes an application of IBFS in a case study, the Magellano project, on well-being measurement in a Public Administration. Finally, we put forth a critical appraisal of our results.

2. FUZZY AND INTUITIONISTIC FUZZY SETS

Let's consider a latent variable, and a universe of discourse X, which implies a qualitative or quantitative interpretation of such variable. For instance, suppose you consider the latent variable ''Effectiveness of University teaching'', and define, as universe of discourse, the set of University courses with respect to which we aim to measure effectiveness. Consider, as another example, the variable ''wealth''; the universe of discourse could be a real value higher than 200,000 euro, corresponding to high annual income.

For a fixed X, consider now the function:

$$\mu(x) \in [0, 1] \tag{1}$$

associating to each $x \in X$ a real number included in $[0, 1]$. The FS is defined as:

$$A = \{x, \mu(x) : x \in X\}$$

where the function $\mu(x)$ measures the degree of membership of x to A. If $\mu(x) = 0$, x does not belong to A; if $\mu(x) = 1$, x certainly belongs to A.

For a fixed universe of discourse X and a subset A, we can also consider a function:

$$v(x) \in [0, 1] \tag{2}$$

such that:

$$0 \leq \mu(x) + v(x) \leq 1, \forall x \in X$$

The set:

$$A_I = \{x, \mu(x), v(x) : x \in X\} \tag{3}$$

is an IFS and the function $v(x)$ measures the degree of non-membership of x to the set A_I. In case the IFS in (3) is composed by only one element u, it is called a singleton and will be indicated with $\alpha = \langle u, \mu, v \rangle$. In this intuitionistic framework, we also introduce the function:

$$\pi(x) = 1 - \mu(x) - v(x) \tag{4}$$

expressing the degree of uncertainty. It is worth observing that a FS can always be interpreted as IFS by defining $v(x) = 1 - \mu(x)$ without invoking the uncertainty parameter. From the one hand, several possibilities to jointly assign functions (1) and (2) through mathematical models have been put forth. From the other hand, it is less common to find proposals to jointly assign functions (2) and (4). It has been suggested (Marasini, Quatto and Ripamonti, 2015) to consider:

$$\pi(x) = \mu(x)^{\eta}[1 - \mu(x)]^{\vartheta} \tag{5}$$

with $\eta, \vartheta \geq 1$, obtaining $v(x)$ through (4). The researcher should choose a suitable function in (5) for $\mu(x)$.

A function well adapted to many real cases is the spline function:

$$\mu(x) = \begin{cases} 0 & \text{if } x < a \\ \frac{1}{2} - \frac{1}{2}\left(2\frac{c-x}{b-a}\right)^2 & \text{if } a \leq x \leq c \\ \frac{1}{2} + \frac{1}{2}\left(2\frac{x-c}{b-a}\right)^2 & \text{if } c < x \leq b \\ 1 & \text{if } x > b \end{cases} \tag{6}$$

with $c = \frac{b+a}{2}$, $\eta = \vartheta = 1$. From (6), we obtain the following uncertainty and non-membership functions:

$$\pi(x) = \mu(x)[1 - \mu(x)] = \begin{cases} \frac{1}{4} - \frac{1}{4}\left(2\frac{c-x}{b-a}\right)^4 & \text{if } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

$$\nu(x) = [1 - \mu(x)]^2 = \begin{cases} 1 & \text{if } x < a \\ \left[\frac{1}{2} + \frac{1}{2}\left(2\frac{c-x}{b-a}\right)^2\right]^2 & \text{if } a \leq x \leq c \\ \left[\frac{1}{2} - \frac{1}{2}\left(2\frac{x-c}{b-a}\right)^2\right]^2 & \text{if } c < x \leq b \\ 0 & \text{if } x > b \end{cases} \tag{8}$$

To illustrate the previous concepts, let's now consider a practical case, in which we focus on the satisfaction expressed by a group of subjects with respect to a certain item. Thus, let's consider a putative satisfaction questionnaire compiled by the employees of a company, where they have been asked to respond to the question: "Is there collaboration among colleagues?". This item can be considered a proxy of well-being in the company. Imagine that this questionnaire is structured with four modalities of response:

Decidedly No (DN); More No than Yes (MN); More Yes than No (MY); Decidedly Yes (DY) (9)

where DN indicates absence of collaboration, thus being interpretable as absence of satisfaction with respect to collaboration; DY expresses full collaboration, thus being interpretable as complete satisfaction. We can transform the four modalities in (9) with the integers 1-4, thus obtaining the responses from a sample of $N = 5$ putative raters, as illustrated in Table 1.

TABLE 1. - *Ratings (indicated with *) given by putative five employees ($i = 1, \dots, 5$) to a question on satisfaction in their company. The scale used to evaluate is composed by the modes: Decidedly No (DN), More no than Yes (MN), More yes than no (MY), Decidedly Yes (DY). We calculated the values $\mu(x) = \mu_i$ and $\nu(x) = \upsilon_i$ using (6) and (8) with $a = 1$ and $b = 4$, quantifying (9) with the first four integers*

i	DN=1	MN=2	MY=3	DY=4	μ_i	ν_i
1	*				0	1
2		*			0.44	0.31
3		*			0.44	0.31
4			*		0.55	0.20
5				*	1	0

The universe of discourse is represented by individuals, indexed by $i, i = 1, \dots, N$; to each one of them it is indeed assigned the pair $(\mu_i, \nu(i) = \nu_i)$ and the IFS in (3) is given by:

$$A_I = \{\langle i, \mu_i, \nu_i \rangle : i = 1, \dots, 5\} \tag{10}$$

In the survey context it is also necessary to synthesize subjects' responses, which, in the intuitionistic context, can be achieved through the Intuitionistic Weighted Mean. (IWAM, Beliakov, Bustince, Goswami, Mukherjee and Pal, 2011) – aggregator, leading to the following pair:

$$IWAM_w(\alpha_1, \dots, \alpha_N) = (\mu = \sum_{i=1}^{N} w_i \mu_i,\ \nu = \sum_{i=1}^{N} w_i \nu_i) \tag{11}$$

with $\alpha_i = \langle i, \mu_i, \nu_i \rangle$; $0 \le w_i \le 1$; $\sum_{i=1}^{N} w_i = 1$. From the example in Table 1, we could interpret each α_i as a singleton, setting $w_i = 1/5$, with $i = 1, \dots, 5$, thus obtaining the pair $(\mu = 0.49;\ \nu = 0.36)$. In an intuitionistic framework, such result can be interpreted stating that this set of five employees, considered as a group, expressed a degree of satisfaction of 0.49, a degree of dissatisfaction of 0.36, with a residual uncertainty of 0.15. Thus, (3) can be expressed through (12):

$$\alpha_I = \langle u, \mu = 0.49, \nu = 0.36 \rangle \tag{12}$$

where u represents the set of five employees jointly considered, so that it should be noted that two different universes of discourse underpin (10) and (12).

An alternative way to quantifying membership and non-membership values is to consider, given a scale monotonically expressing satisfaction (such as scale (9)), a dichotomous definition. We will assign value 1 to a putative participant, only if he/she is supposed to rate the superior value of the scale (i.e., "Decidedly yes" in scale

(9)). In other cases, we'll assign value 0. This definition underpins a concept of membership in the sense of excellence; in fact, only if a participant expresses full satisfaction the researcher will assign a membership value, and this value is 1 (full satisfaction). In a similar manner, we'll assign value 1 to non-membership, only if the participant has provided response in correspondence to the lower end of the scale (i.e., ''Decidedly no'' in scale (9)); otherwise it will be assigned value 0.

Formally, given a monotonous scale with j modalities, let's indicate with a and b the lower and upper bound, respectively, and let c and d be two internal threshold of this scale. For each subject, the values of membership and non-membership, defined by (3), where the universe of discourse refers to the degree of satisfaction, can be assigned as follows:

$$\begin{aligned} \mu(x) &= \begin{cases} 0 & \text{if } a \le x < c \\ 1 & \text{if } c \le x \le b \end{cases} \\ \nu(x) &= \begin{cases} 1 & \text{if } a \le x \le d \\ 0 & \text{if } d < x \le b \end{cases} \end{aligned} \tag{13}$$

With regard to scale (9), we can fix $\alpha = d = 1$ and $b = c = 4$ and, considering the putative ratings in Table 1, applying (13), we obtain Table 2.

TABLE 2. - *Results obtained from the same putative five employees presented in Table 1, using (13)*

I	**DN=1**	**MN=2**	**MY=3**	**DY=4**	μ_i	ν_i
1	*				0	1
2		*			0	0
3		*			0	0
4			*		0	0
5				*	1	0

To synthetize participants' ratings, we now re-define the universe of discourse as satisfaction with respect to the item expressed by the sample as a whole (previously indicated by u) and, through (11) with weights $\omega_i = 1/5$, we obtain the singleton

$$\alpha_F = \langle u, \mu = 0.20, \nu = 0.20 \rangle$$

We can observe that membership and non-membership, defined through (13) and subsequently using (11) (hence obtaining the singleton α_F) coincide with the relative frequencies calculated in the sample of respondents, with respect to the superior modality (''Decidedly yes'') and inferior modality (''Decidedly no''). The corresponding uncertainty can be obtained through (4), in our example $\pi = 0.60$, which indicates a substantial amount of uncertainty.

3. THE BIPOLAR MEAN

The bipolar mean can be used as a measure of synthesis in case of ordinal variables. Let's consider:

$$x_{(1)}, \ldots, x_{(j)}, \ldots, x_{(k)} \tag{14}$$

with k ordered modalities of a qualitative variable. Let $(p_1, \ldots, p_j, \ldots, p_k)$ be the correspondent relative frequencies, with $\sum_{j=1}^{k} p_j = 1$, where, for the sake of simplicity, we put $p_j = p(x_{(j)})$. Let R_j be the relative retro-cumulative frequency in correspondence of $x_{(j)}$, i.e., $R_j = \sum_{i=j}^{k} p_i$.

If $(p'_1, \ldots, p'_j, \ldots, p'_k)$ are the frequencies associated to another distribution of the same variable, and $(R'_1, \ldots, R'_j, \ldots, R'_k)$ are the respective retro-cumulative frequencies, a distribution $(p_1, \ldots, p_j, \ldots, p_k)$ is said to dominate $(p'_1, \ldots, p'_j, \ldots, p'_k)$ if $R'_j \leq R_j$, for each $j = 1, \ldots, k$, with, at least, one strict inequality. The bipolar mean is given by:

$$s = \sum_{j=1}^{k} R_j \tag{15}$$

which is an arithmetic mean, can be ordered and maintains the original ordering structure, so that if $(p_1, \ldots, p_j, \ldots, p_k)$ dominates $(p'_1, \ldots, p'_j, \ldots, p'_k)$, then $s(p_1, \ldots, p_j, \ldots, p_k) \geq s(p'_1, \ldots, p'_j, \ldots, p'_k)$.

The bipolar mean can indeed be conceived as a synthesis allowing to transform a given distribution in another theoretical distribution, which either associates 1 to only one modality, or subdivides the unit between two contiguous modes. In particular:

1. If s coincides with the integer $j (1 \leq j \leq k)$ corresponding with the j –th position in (14), the new distribution concentrates 1 on the modality $x_{(j)}$;
2. If $j < s < j + 1$, the new distribution subdivides the unit between the modalities $x_{(j)}$ and $x_{(j+1)}$, so that to $x_{(j)}$ is associated the relative frequency $j + 1 - s$ and to $x_{(j+1)}$ the relative frequency $s - j$.

It can be shown that:

$$s = \sum_{j=1}^{k} R_j = \sum_{j=1}^{k} jp_j \tag{16}$$

which can be conceived as a weighted arithmetic mean, so that, if to the k modalities in (14) we substitute the k integers $1, \ldots, j, \ldots, k$, (16) coincides with the mean of the distribution $(1,\ p_1), \ldots, (j,\ p_j), \ldots, (k,\ p_k)$. If $s = j$, then to the original distribution it is associated the bipolar distribution $(1, p^*_1 = 0), \ldots, (j, p^*_j = 1), \ldots, (k, p^*_k = 0)$; if else $j < s < j + 1$, to the original distribution it is associated the bipolar distribution $(1, p^*_1 = 0), \ldots, (j, p^*_j = j + 1 - s), (j + 1, p^*_{j+1} = s - j), \ldots, (k, p_k = 0)$.

For the sake of illustration, let's consider again the previous question: "Is there collaboration among colleagues?", from a putative satisfaction questionnaire. Summing the retro-cumulative frequencies in Table 3, we obtain $s = 2.4$ and the implied bipolar distribution.

TABLE 3. - *Bipolar distribution for the motivating example*

X	p_j	R_j	p_j^*
DN	0.20	1.00	0.00
MN	0.40	0.80	0.60
MY	0.20	0.40	0.40
DY	0.20	0.20	0.00

From this Table, following the proponents of the bipolar means, we could conclude that 60% of the participants are not so much satisfied of their colleagues, whereas 40% of the participants are enough satisfied. In the present context, we'll present a different perspective on bipolar distributions, which is not aimed at directly interpreting satisfaction, but instead at quantifying uncertainty.

It has been shown that, fixed the size N of a certain population and the number k of ordered modalities, there do exist $\binom{N+k-1}{N}$ distinct distributions, leading to $\gamma = N(k-1)+1$ bipolar means of type (15), taking values:

$$1, \frac{N+1}{N}, \ldots, \frac{N+t}{N}, \ldots, k \tag{17}$$

Let's indicate with B the set composed by each possible distribution; this can be subdivided in γ equivalence classes, each of which is made up by all possible distributions leading to the same bipolar mean, the r-th of which is represented by a bipolar distribution.

Finally, we define the following relation: consider a function $f(\cdot)$, such that $\boldsymbol{B} = f(B)$; $\boldsymbol{B}$ is made up by a single equivalence class constituted by each distribution guaranteeing the following bipolar mean:

$$s = \sum_{r=1}^{\gamma} \omega_r s_r \tag{18}$$

with $\omega_r = \frac{1}{H}$, and H bipolar means of B considered $(H \leq \gamma)$, and $\omega_r = 0$ for the others.

4. BIPOLAR DISTRIBUTIONS IN FUZZY SETS THEORY

Fuzzy theory is, in our view, a powerful tool to interpret evaluation questionnaires. Nevertheless, the fuzzy approach on satisfaction emphasizes positive responses. In fact, ‘‘the fuzzy set is not able to express the neutral state, i.e., neither supporting nor opposing’’ (Li, 2014). By contrast, the intuitionistic approach allows to emphasize both positive evaluations and their negative counterpart, thus being more useful in performance evaluation. In other words, ‘‘the intuitionistic fuzzy set is able to describe fuzziness of neither this nor that’’ (Li, 2014). However, the intuitionistic framework considers the uncertainty as a whole, and does not allow to distinguishing between a positive and a negative component. This can be a serious limitation in order to properly interpret questionnaire results. The bipolar distribution approach allows a researcher to handle such a distinction, and can be integrated with intuitionistic fuzzy sets.

Evaluation questionnaires envisage different latent variables, quantified by items that can be rated by respondents using a qualitative or quantitative scale. In (9) we introduced a qualitative ordinal scale, which can be quantified by the first four integers or with other proposals.

Scale (9) has been currently adopted by the Italian Ministry of Public Administration in evaluation surveys. The same scale is also used by the Ministry of Education to evaluate students' satisfaction for University courses. Other two frequently used 5-ponts scales are:

Strongly disagree, Disagree, Uncertain, Agree, Strongly agree (19)

Very insufficient, Insufficient, Sufficient, Good, Very good (20)

which can be quantified by the first five integers. Scale (19) has been frequently used in customer satisfaction surveys. Recently, in Italy, the Public Anti-Corruption Authority (Autorità Nazionale Anticorruzione, ANAC) has introduced another 6-points scale, ranging from ‘‘Definitely Disagree’’ to ‘‘Definitely Agree’’:

‘‘Definitely disagree’’, ‘‘Disagree’’, ‘‘Partially disagree’’, ‘‘Partially agree’’, ‘‘Agree’’, ‘‘Definitely agree’’

This ANAC scale and scale (9) have in common that they contain very sharp and well-defined categories of response, such as ‘‘Strongly agree’’ or ‘‘Definitely disagree’’, but also more nuanced and fuzzy modes, such as ‘‘More no than yes’’ or ‘‘More yes than no’’. In the present context, we propose a method to analyze such item-related component of uncertainty. We aim indeed at differentiating between a positive and a negative source of uncertainty. Scale (9) is entirely suitable for a bipolar fuzzy analysis, with the two modes ‘‘More yes than no’’ and ‘‘More no than yes’’, naturally related to a positive and negative source of uncertainty, respectively. Also the 6-points ANAC scale is particularly indicated for a bipolar fuzzy analysis.

Differently, in other scales, such as (19) and (20), it is more difficult to clearly distinguish between a positive and a negative source of uncertainty. In these scales each modality seems to be clearly defined, and it is more probable to occur a source of uncertainty related to subject's choices rather than to items' modalities. Other statistical models focus on the uncertainty related to subjects, such as the Combined

Uniform-Binomial model (D'Elia and Piccolo, 2005), which has been currently employed in several contexts.

To deal with item-related uncertainty, we'll now introduce a new kind of Intuitionistic Fuzzy Sets, which will be indicated in the following as Intuitionistic Bipolar Fuzzy Sets (IBFS). To the sake of illustration, we'll consider the bipolar distribution presented in Table 3. Basically, our proposal is to interpret the values assigned to the modalities "More no than yes" (MN) and "More yes than no" (MY) as a proxy of negative $(\pi^- = 0.60)$ and positive $(\pi^+ = 0.40)$ uncertainty, respectively.

More in general, considering (17) with $k = 4$, it follows that the bipolar mean takes values between 1 and 4, hence to the four modalities DN, MN, MY, DY can be associated the values in Table 4, where, to each row, it is associated a bipolar distribution for different values of s.

TABLE 4. - *Hypothetical Table to define IBFS in case of a questionnaire using a scale such as (9); each row represents a possible bipolar distribution for different values of* s

	$DN = 1$	$MN = 2$	$MY = 3$	$DY = 4$
$1 \leq s \leq 2$	2-s	s-1	0	0
$2 \leq s \leq 3$	0	$3 - s$	s-2	0
$3 \leq s \leq 4$	0	0	4-s	s-3

Table 4 shows that the bipolar technique, following different values of s, can directly assign values to the pairs (DN, MN), (MN, MY), (MY, DY); we particularly focus on the pair (MN MY), indicated with (π^-, π^+), which quantifies the source and the direction of uncertainty. This pair can be directly obtained, for $2 \leq s \leq 3$, such as in our previous example, where $s = 2.4$. However, the intuitionistic pair (MN, MY) can also be easily obtained indirectly. For the sake of illustration, let's now imagine that $s = 1.8$: from the bipolar distribution it would follow the pair $(\mathrm{DN} = 0.2, \mathrm{MN} = 0.8)$, hence with the bipolar fuzzy analysis we would focus on the pair $(\pi^- = 0.8, \pi^+ = 0)$. In the extreme case scenario, if we had obtained either $s = 1$ or $s = 4$, it would have followed the pair $(\pi^- = 0, \pi^+ = 0)$, thus indicating absence of uncertainty.

Interpreting the values (π^-, π^+) as measures of negative and positive uncertainty, we can introduce an Intuitionistic Bipolar Fuzzy Set given by:

$$\alpha_B = \langle u, \pi^-, \pi^+ \rangle \tag{21}$$

which, in our previous motivating example, would be the singleton $\alpha_B = \langle u, \pi^- = 0.60, \pi^+ = 0.40 \rangle$. Sets in (21) can be conceived as a special case of IFS, where, fixed a bipolar mean, three parameter values are assigned: (i) the decomposition parameter (π^-, π^+); (ii) either μ or ν, and (iii) the missing parameter, indirectly, with a simple extension of the intuitionistic constraint.

Table 4 can be easily extended to 6 modalities (e.g., the ANAC scale), and a bipolar intuitionistic fuzzy analysis can be proposed also in this case. In particular, the pair ''Partially disagree'' and ''Partially agree'' can be conceived as the pair (π^-, π^+). With parameters π^- and π^+ fixed, the remaining negative modalities could be potentially merged, as well as the remaining positive modalities. If $1 \leq s \leq 2$, only the negative component is quantified and there is no uncertainty; if $2 \leq s \leq 3$ both the negative component and negative uncertainty are quantified; if $3 \leq s \leq 4$ only the uncertainty (negative and positive) takes positive values; if $4 \leq s \leq 5$ positive uncertainty and the positive component are quantified; if else $5 \leq s \leq 6$ there is no uncertainty (see Table 5).

IBFS could be naturally taken also to quantifying uncertainty with other scales and with more than 6 modalities. This can be easily obtained by extending Table 5.

TABLE 5. - *Hypothetical Table to define IBFS in the case of the 1-6 ANAC scale; each row represents a possible bipolar distribution for different values of* s

	1	2	3	4	5	6
$1 \leq$ s ≤ 2	2-s	s-1	0	0	0	0
$2 \leq s \leq 3$	0	3-s	s-2		0	0
$3 \leq s \leq 4$	0	0	4-s	s-3	0	0
$4 \leq s \leq 5$	0	0	0	5-s	s-4	0
$5 \leq s \leq 6$	0	0	0	0	6-s	s-5

5. AN APPLICATION OF IBFS: THE MAGELLANO PROJECT

We propose an application of IBFS to a national questionnaire prepared by the Italian Ministry of Public Administration, which is distributed to employees and managers of Public Administrations, and is designed to measure their well-being and satisfaction (*The Magellano* Project, http://magellanopa.it). This survey is dedicated to monitoring the opinions of the administrative staff, in order to ameliorate the quality of the work environment, the effectiveness, and the performance of the Institution. The questionnaires are compiled online and are composed by 48 questions, for an average compile-time of 15/20 minutes. Overall, the questionnaire is structured in 12 thematic areas; these are preceded by the detection of information regarding the respondents (gender, age, area of origin, staff position...). In general, the proposed items have four modes of response with the addition of a field indicated with ''other, do not know''. The confidentiality of the collected information is guaranteed by the local Evaluation Committee of each single Institution. A communication, in which it is provided information on the survey and on how to complete the questionnaire, is sent by e-mail to all the staff of Public administrations (e.g., Universities, Government Departments). The following thematic areas, leading to latent variables, have been considered in the questionnaire: (i) The characteristics of the work environment; (ii) The perceived safety; (iii) The relationship with colleagues; (iv) The efficiency; (v) The organization and the fairness of the Administration; (vi) The condition and the quality of work; (vii) The management; (viii) The characteristics of the work; (ix) Positive and negative indicators of organizational well-being; (x) The mental and physical well-being; (xi) Openness to innovation; (xii) Further suggestions.

We considered data from one Public Institution, with $N = 210$ respondents. In particular, we focus on four latent variables: Fairness of the organization, Management, Efficiency, and Openness of the Administration to innovation. Details on the items used to measure these latent variables are given in the Appendix. Responses marked as ''other, do not know'' have been deleted from the following analyses (less than 5% of the entire sample).

Before applying IBFS, we report IFS obtained either using models (6) and (8), or adopting (13) and the corresponding relative frequencies. To obtain a synthesis by subjects and by items, the aggregator (11) is used twice: firstly aggregating by subjects (on each item), and secondly aggregating more items measuring a same latent variable. In this way, in our sample of respondents, we obtained, using the mathematical models in (6) and (8):

$$\alpha_I = \langle \textit{Fairness}, \mu = 0.31, \nu = 0.55 \rangle$$
$$\alpha_I = \langle \textit{Managers}, \mu = 0.51, \nu = 0.33 \rangle$$
$$\alpha_I = \langle \textit{Efficiency}, \mu = 0.55, \nu = 0.28 \rangle$$
$$\alpha_I = \langle \textit{Openness to innovation}, \mu = 0.45, \nu = 0.37 \rangle$$

The uncertainty in each of these cases takes values $\pi = 0.14$, $\pi = 0.16$, $\pi = 0.17$, $\pi = 0.18$, which can be calculated through (4).

Using (13) and relative frequencies to measure membership and non-membership, we obtained:

$$\alpha_F = \langle \textit{Fairness}, \mu = 0.04, \nu = 0.39 \rangle$$
$$\alpha_F = \langle \textit{Managers}, \mu = 0.17, \nu = 0.17 \rangle$$
$$\alpha_F = \langle \textit{Efficiency}, \mu = 0.20, \nu = 0.11 \rangle$$
$$\alpha_F = \langle \textit{Openness to innovation}, \mu = 0.08, \nu = 0.17 \rangle$$

with uncertainty given by $\pi = 0.57$, $\pi = 0.66$, $\pi = 0.69$, $\pi = 0.75$, respectively. The degree of uncertainty, higher in this case than using models (6) and (8) is directly justified by (13).

Let's now consider, for the sake of illustration, the latent variable 1., *Fairness of organization*, with the objective of decomposing uncertainty in a positive and a negative component. In our sample of respondents we obtained, for the four items, the following IBFS:

$$A_B = \left\{ \langle i, \pi_i^-, \pi_i^+ \rangle : i = 1, \ldots, 4 \right\}$$

where the single elements are:

$\alpha_{B1} = \langle 1, \pi^- = 0.67, \pi^+ = 0 \rangle$, $\alpha_{B2} = \langle 2, \pi^- = 0.78, \pi^+ = 0 \rangle$,
$\alpha_{B3} = \langle 3, \pi^- = 0.81, \pi^+ = 0.19 \rangle$, $\alpha_{B4} = \langle 4, \pi^- = 0.83, \pi^+ = 0 \rangle$

with bipolar means: 1.67; 1.78; 2.19; 1.83, respectively.

Such values can be synthetized by a singleton, indicating the uncertainty underlying the variable Fairness of organization, as expressed by the sample of 210 respondents. By means of (18) it is indeed possible to combine the four bipolar means obtained for each item, thus leading to an aggregated intuitionistic fuzzy set. In fact, fixed $\omega_r = 1/4$ for $r = 1, 2, 3, 4$ and $\omega_r = 0$ for $r > 4$, we obtain:

$$s = \sum_{r=1}^{631} \omega_r s_r = 1.87 \tag{22}$$

The singleton expressing uncertainty related to the latent variable 1. is given by:

$$\alpha_B = \langle \mathit{Fairness}, \pi^- = 0.87, \pi^+ = 0 \rangle \tag{23}$$

which indicates prevalence of negative uncertainty. It is worth observing that, amongst 631 potential equivalence classes, only the four representing the four bipolar means are to be considered; consequently the set **B**, the image of B through (22), which interprets the latent variable, is constituted by the only equivalence class characterized by the bipolar distribution: ‘‘DN = 0.13; MN = 0.87; MY = 0, DY = 0’’, leading to (23).

As to the latent variables 2, 3 and 4, using the same procedure we obtained the singletons:

$$\begin{aligned} &\alpha_B = \langle \mathit{Managers}, \pi^- = 0.38, \pi^+ = 0.62 \rangle \\ &\alpha_B = \langle \mathit{Efficiency}, \pi^- = 0.29, \pi^+ = 0.71 \rangle \\ &\alpha_B = \langle \mathit{Openness\ to\ innovation}, \pi^- = 0.71, \pi^+ = 0.29 \rangle \end{aligned} \tag{24}$$

From our application, it emerges that a high number of ratings is concentrated across the two central values of scale (9). In intuitionistic fuzzy terms, this reflects high levels of uncertainty, and should properly subdivided into negative and positive components for a correct understanding of questionnaires’ results. IBFS allows us to conduct such analysis and are very easily interpretable also for policy and decision makers.

6. DISCUSSION

The fuzzy analysis, in our view, represents a valid and very informative instrument for the analysis of survey data. In particular, the intuitionistic framework allows a researcher to quantify the degree of uncertainty related to the scale adopted by the questionnaire. In this paper, we presented a new approach, founded on the descriptive concepts of bipolar means and bipolar distributions, which can be related to IFS to differentiate between a positive and a negative component of uncertainty. Our proposal is indeed to present A_I (IFS) together with A_B (IBFS) for a deeper understanding of questionnaire results. A researcher might choose to use A_F instead of A_I; this is a subjective choice but IBFS can be used in both cases.

In our view IBFS should be considered as a measure of dispersion of the judgments related to satisfaction. Consider, for instance, an item of a questionnaire with respect to which subjects expressed a degree of satisfaction quantified, through IFS, by $\mu = 0.50$. This indicates a very high degree of satisfaction but, without other information, its interpretation can be misleading. Let's fix $\mu = 0.10$, so that, by the intuitionistic constraint, $\pi = 0.40$. Also the uncertainty is quite high, but if it is not decomposed between its positive component and its negative counterpart, it may be difficult to interpret. In fact, from the one hand, a high positive component would strengthen the positive result; from the other hand, a high negative component would underpin a high degree of variability in the ratings, thus leading a researcher to consider with caution the degree of satisfaction. As previously observed, the bipolar mean is indeed a weighted arithmetic mean; nevertheless, we would not recommend to directly use it as a descriptive index in the evaluation context. We would suggest instead to using the bipolar distribution to calculate IBFS, thus obtaining an indicator of the variability of the responses.

It should be noted that IBFS can be also used as a complementary technique of other statistical models for survey data, such as Rasch analysis, CUB models, and Bayesian networks. In particular, CUB models focus on a component of uncertainty linked to subjects, whereas IBFS capture a component of uncertainty related to items. These techniques could be both used in survey data analysis, for a better understanding of questionnaires' results.

To show the potential advantage of IBFS, we now briefly comment the main results obtained in our application. As to *Fairness of organization*, we obtained a very low value of membership along with an intermediate level of non-membership (i.e., dissatisfaction). The average degree of satisfaction is estimated to be very low. This information should urgently lead a Public Administration to adopt measures aimed to obtain a higher degree of Fairness.

A slightly better result has been obtained with *Management*. In this case it has been calculated an average degree of satisfaction, with a prevalent degree of positive uncertainty. This result should be considered with caution, but it indicates that the Public Administration has a good Management. Similar results have been obtained in the case of the latent variable *Efficiency*.

A negative result has been obtained with respect *Openness to innovation*; the degree of membership proved to be very low, with a quite high degree of negative uncertainty. Also in this case, this Public Administration should put forward a specific policy to ameliorate its performance.

In conclusion, from this analysis it emerges that IBFS allow a decision maker to focus not only on the average performance achieved by the Administration, but also with the related degree of uncertainty and variability. Should a high performance in terms of satisfaction be considered with caution, in case the respondents expressed a high degree of uncertainty? Our analysis allows policy makers to address this question in a relatively simple and straightforward manner.

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APPENDIX

Items of the *Magellano* questionnaire considered in our application

1. *Fairness of organization*: (a) the Administration offers equal career opportunities to all its employees; (b) the commitment at work and personal initiatives are appreciated (with financial rewards, praise,...); (c) the management offers opportunities for professional development; (d) economic incentives are distributed on the basis of effectiveness of performance.

2. *Management*: (a) managers behave fairly with employees; (b) managers are coherent with the objectives and goals of the organization; (c) managers want to be informed on problems and difficulties at work; (d) managers involve employees in strategic decisions; (e) the criteria for employees' evaluation are fair and transparent.

3. *Efficiency*: (a) the objectives of the organization are clear and well-defined; (b) the organization owns means and resources to carry out its work properly; (c) the organization can find adequate solution to challenging problems; (d) at the end of the working day you feel satisfied; (e) work allows to bring out personal and professional qualities of each employee; (f) when you need information, you know how to obtain them; (g) organizational roles and work tasks are clear and well-defined.

4. *Openness to innovation*: (a) the Administration is careful to acquire new technologies to be used in the work; (b) the Administration puts attention in new working processes; (c) the organization is careful to compare the experience of other administrations; (d) the organization is careful in recognizing and properly addressing past problems and mistakes; (e) the organization is careful in accommodating users' requests; (f) the organization is careful in introducing new skills; (g) the organization is careful to develop innovative skills in its employees; (h) the organization is careful in establishing collaborative relations with other companies; (i) the organization aims at experimenting new forms of organization of work.

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