Statistica & Applicazioni - Vol. XVII, n. 1, 2019, pp. 3-33

ASYMPTOTIC CONFIDENCE INTERVALS FOR PARAMETERS ESTIMATED THROUGH THE RATIO OF ASYMPTOTICALLY NORMAL STATISTICS

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SUMMARY

In this paper, four different approaches for the definition of asymptotic confidence intervals for the ratio of two unknown parameters are reviewed and compared via a simulation study. The considered approaches are based on the well known Delta Method and on the distribution of the ratio of correlated normal random variables. Simulations concern the ratio between two expectations, the Coefficient of Variation, the Gini Concentration Ratio, and the Sharpe Ratio. It is shown that the asymptotic confidence intervals based on the ratio of correlated normal random variables often have a better coverage accuracy with respect to the ones derived from Delta Method, even if the observed gain is small in some cases.

Keywords: ratio of correlated normal random variables, Delta Method, Gini Concentration Ratio, Coefficient of Variation, Sharpe Ratio.
DOI: 10.26350/999999_000017
ISSN: 18246672 (print)

1. INTRODUCTION

In various practical statistical analysis, the phenomena of interest is studied through indicators defined by the ratio of two parameters. The Gini Concentration Ratio in the study on income inequality (see, e.g., Zenga, 2007), the Sharpe Ratio in the evaluation of financial performance (see Sharpe, 1964, 1966, 1994), and the Cost-Effectiveness ratio often used in the evaluation of medical treatment (see Galeone and Pollastri, 2012), the Coefficient of Variation are just few examples. These indicators are usually estimated by means of a ratio of two statistics which are jointly asymptotically normally distributed. In this case, an asymptotic Confidence Interval (CI) for the indicator is often obtained thanks to the well-known ''Delta Method'' (see, e.g., Serfling, 1980, or Casella and Berger, 2002, Section 5.5.4). However, as suggested by Galeone (2007), Galeone and Pollastri (2012, 2013) and Pollastri and Maffenini (2018), such a CI can be obtained also by using the distribution of the ratio of two Correlated Normal Random Variables (CNRV). Indeed, when sampling

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from a bivariate normal distribution, these authors showed that the CI for the ratio of means obtained by the last technique, have an actual coverage probability which is often closer to the nominal one with respect to CIs obtained with standard techniques, such as the Fieller's method (see Fieller, 1932, 1940, 1954). In this paper, we review the literature on the distribution of the ratio of two CNRV and we deepen the evaluation of the performance of CIs based that distribution. Moreover, the absolute value of the ratio (Pollastri and Tulli, 2015) and the truncated distribution of the ratio are considered to attempt improving the performance of the related CIs when the indicator of interest is positive or takes value in a bounded interval.

The paper is organized as follows. In Section 2 the relevant literature on the distribution of the ratio of CNRV is reviewed. In Section 3 the possible approximations of the distribution of the ratio are discussed to provide the theoretical bases for the introduction of the asymptotic CIs considered in the paper. Section 4 concerns the truncated distribution of the ratio of CNRV and the distribution of the absolute value of the ratio. In Section 5 different CIs for the ratio are introduced and, in Section 6, their coverage accuracy is compared via a simulation study. The indicators considered in the simulations are: the ratio of the expected values in a bivariate non-Normal random variable; the Gini Concentration Ratio, defined as the ratio of the Gini mean difference and two times the expected value of the random variable under study; the Coefficient of Variation, defined as the ratio of the standard deviation and the expected value of a random variable; the Sharpe Ratio, defined as the ratio of the expected value and the standard deviation of the random variable describing the excess return of a risky financial activity. Simulations related to the Sharpe Ratio are performed under a time series setting while the remaining ones are performed assuming simple random sampling. Finally, Section 7 is devoted to conclusions. An appendix with the description of the `R`-function useful to compute the considered CIs is also given.

2. THE DISTRIBUTION OF THE RATIO OF TWO CORRELATED NORMAL RANDOM VARIABLES

The distribution of the ratio of two CNRV has been studied by many authors. Geary (1930) was the first who considered the ratio of two CNRV and found an approximation of its distribution. Later, Fieller (1932, 1940, 1954) found the distribution function of the ratio of CNRV and introduced approximate confidence intervals for the indicator starting from an approximation similar to the one in Geary (1930). Specifically, Fieller (1940) provided the expression of an approximate confidence interval for the ratio of the means of a Bivariate Normal Distribution. Hinkley (1969) and Marsaglia (1965) contributed to improve the studies on the subject. For a good review and further insights on this topic, see Galeone (2007) and Marsaglia (2006). More recently, Aroian (1986) and Öksoy and Aroian (1986) proposed a new formulation of the probability density function (pdf) of the ratio of CNRV. We recall that formulation and many others, below.

Let us consider a Bivariate Normal (BN) random variable

$$\begin{bmatrix} Y \\ X \end{bmatrix} \sim \mathbf{N}\left(\begin{bmatrix} \mu_Y \\ \mu_X \end{bmatrix}, \begin{bmatrix} \sigma_Y^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{bmatrix} \right)$$

where

$$\mu_Y = E[Y]\,,\ \ \mu_X = E[X]\,,\ \ \sigma_Y^2 = E[(Y-\mu_Y)^2]\,,$$

$$\sigma_X^2 = E[(X-\mu_X)^2]\,,\ \text{and}\ \rho = \frac{E[(Y-\mu_Y)(X-\mu_Y)]}{\sigma_Y\sigma_X}\,.$$

In Öksoy and Aroian (1986) it is shown that the cumulative distribution function (cdf) of the random variable $W = \frac{Y}{X}$ is given by

$$F_W(w) = L\left(\frac{a - bt_w}{\sqrt{1+t_w^2}}, -b, \frac{t_w}{\sqrt{1+t_w^2}} \right) + L\left(\frac{bt_w - a}{\sqrt{1+t_w^2}}, b, \frac{t_w}{\sqrt{1+t_w^2}} \right) \tag{1}$$

where $w \in \mathbb{R}$,

$$a = \sqrt{\frac{1}{1-\rho^2}} \left(\frac{\mu_Y}{\sigma_Y} - \rho\frac{\mu_X}{\sigma_X} \right), \qquad b = \left(\frac{\mu_X}{\sigma_X} \right), \tag{2}$$

$$t_w = \sqrt{\frac{1}{1-\rho^2}} \left(\frac{\sigma_X}{\sigma_Y} w - \rho \right) \tag{3}$$

and $L(h,k,p)$ is the bivariate normal integral according to the definition of Kotz, Balakrishnan and Johnson (2004):

$$L(h,k,p) = \left(2\pi\sqrt{1-\rho^2}\right)^{-1} \int_h^\infty \int_k^\infty \exp\left(-\frac{1}{2(1-\rho^2)}(u^2 - 2\rho uv + v^2) \right) dvdu\,.$$

A further expression for F_W was derived in Öksoy and Aroian (1994):

$$F_W(w) = \frac{1}{2} + \frac{1}{\pi}\arctan(t_w) + 2V\left(\frac{bt_w - a}{\sqrt{1+t_w^2}}, \frac{b + at_w}{\sqrt{1+t_w^2}} \right) - 2V(b,a)\,, \tag{4}$$

where $V(h,q)$ is the Nicholson function (Nicholson, 1943, and Sheppard, 1900)

$$V(h,q) = \int_0^h \Phi(u) \int_0^{q\frac{u}{h}} \Phi(v) dvdu$$

and $\Phi(\cdot)$ is the cdf of a standard normal random variable. Starting from (4) and the definition of the T function introduced in Owen (1956)

$$T(h,\lambda) = \frac{1}{2\pi}\arctan(\lambda) - V(h,\lambda h)\,,$$

Pollastri and Tulli (2015) derived the following alternative formula for F_W:

$$F_W(w) = \frac{1}{2} + \frac{1}{\pi} \arctan(t_w) + \frac{1}{\pi} \arctan\left(\frac{b + at_w}{bt_w - a}\right) + \\ - 2T\left(\frac{bt_w - a}{\sqrt{1 + t_w^2}}, \frac{b + at_w}{bt_w - a}\right) - \frac{1}{\pi} \arctan\left(\frac{a}{b}\right) + 2T\left(b, \frac{a}{b}\right). \quad (5)$$

Concerning the pdf $f_W(\cdot)$ of W, in Marsaglia (1965, 2006) and Aroian (1986) it is obtained that:

$$f_W(w) = \frac{\sigma_X}{\sigma_Y} \frac{1}{\sqrt{1 - \rho^2}} f(t_w) \quad (6)$$

where

$$f(x) = \frac{e^{-\frac{1}{2}(a^2+b^2)}}{\pi(1 + x^2)} \left[1 + qe^{\frac{1}{2}q^2} \int_0^q e^{-\frac{1}{2}v^2} dv\right] \\ = \frac{e^{-\frac{1}{2}(a^2+b^2)}}{\pi(1 + x^2)} \left[1 + qe^{\frac{1}{2}q^2} \sqrt{2\pi}\left(\Phi(q) - \frac{1}{2}\right)\right] \quad (7)$$

and

$$q = \frac{b + ax}{\sqrt{1 + x^2}} .$$

From the relation between $f_W(\cdot)$ and $f(\cdot)$ it emerges that W can be viewed as a linear transformation of a random variable W^* with density $f(\cdot)$. Specifically, Marsaglia (1965, 2006) showed that

$$W \stackrel{d}{=} \frac{1}{r} W^* + s$$

where $\stackrel{d}{=}$ means ‘‘equality in distribution’’,

$$r = \frac{\sigma_X}{\sigma_Y \sqrt{1 - \rho^2}} , \quad s = \rho \frac{\sigma_Y}{\sigma_X} ,$$

and the random variable W^* admits the following representation:

$$W^* = \frac{a + H}{b + K} \quad \text{with } H \text{ and } K \text{ independent standard normal.}$$

As shown in (6) and (7), the behavior of $f_W(\cdot)$ can be obtained from the one of $f(\cdot)$, that can be represented as a mixture:

$$f(x) = e^{-\frac{1}{2}(a^2+b^2)} f_1(x) + \left(1 - e^{-\frac{1}{2}(a^2+b^2)}\right) f_2(x) \\ = pf_1(x) + (1 - p)f_2(x) \quad (8)$$

where $p = e^{-\frac{1}{2}(a^2+b^2)}$, $f_1(x) = [\pi(1+x^2)]^{-1}$ is the pdf of a Cauchy random variable and

$$f_2(x) = \frac{q \int_0^q e^{-\frac{1}{2}(v^2-q^2)} dv}{\pi(1+x^2)\left(e^{\frac{1}{2}(a^2+b^2)} - 1\right)} .$$

Note that the ''Cauchy component'' $f_1(\cdot)$ in the mixture does not depend on the parameters of the bivariate normal distribution: apart from limiting cases, the distribution of the ratio of CNRV always has a ''Cauchy component''. This implies that W does not have moments of all orders (apart from limiting cases that will be described next). Moreover, note that the component $f_2(\cdot)$ does not play any role in determining the shape of $f_W(\cdot)$ if and only if $\mu_X = \mu_Y = 0$.

3. APPROXIMATIONS OF F_W AND f_W

In this section we recall and discuss the problem of approximating the distribution of W. Indeed, to evaluate $F_W(\cdot)$ it is necessary to numerically solve some integrals. Some decades ago, these numerical analyses were difficult and some approximations were necessary to easily perform computations. Nowadays, the power of modern calculators eliminates these difficulties and the approximations of $F_W(\cdot)$ might seem useless. However, as it will be clear later, they are very important to provide a theoretical justification of the CIs we will introduce next and to make the connection among the different CIs clear.

In order to introduce an approximation of the distribution of W, observe that

$$\begin{aligned} P\left(\frac{Y}{X} \le w\right) &= P\left(\frac{Y}{X} \le w | X < 0\right) P(X < 0) + \\ &\quad + P\left(\frac{Y}{X} \le w | X > 0\right) P(X > 0) \\ &= P\left(\frac{Y}{X} \le w \cap X < 0\right) + P\left(\frac{Y}{X} \le w \cap X > 0\right) \end{aligned} \tag{9}$$

As shown in Geary (1930) and Frosini (1970), if μ_X is positive and huge with respect to σ_X ($\mu_X >> \sigma_X$) then

$$P\left(\frac{Y}{X} \le w \cap X < 0\right) \approx 0$$

since $P(X < 0) \approx 0$ and

$$\begin{aligned} P\left(\frac{Y}{X} \le w\right) &\approx P\left(\frac{Y}{X} \le w | X > 0\right) P(X > 0) \\ &\approx P\left(\frac{Y}{X} \le w | X > 0\right) \cdot 1 \\ &\approx P(Y \le wX) = P(Y - wX \le 0) . \end{aligned} \tag{10}$$

The last member in (10) leads to

$$F_W(w) \approx \tilde{F}_W(w) = \Phi\left(\frac{w\mu_X - \mu_Y}{\sqrt{\sigma_Y^2 - 2w\rho\sigma_X\sigma_Y + w^2\sigma_X^2}}\right) \tag{11}$$

and

$$f_W(w) \approx \tilde{f}_W(w) = \phi\left(\frac{w\mu_X - \mu_Y}{\sqrt{\sigma_Y^2 - 2w\rho\sigma_X\sigma_Y + w^2\sigma_X^2}}\right) H(w, \mu_Y, \mu_Y, \sigma_X, \sigma_Y, \rho) \tag{12}$$

with

$$H(w, \mu_Y, \mu_Y, \sigma_X, \sigma_Y, \rho) = \frac{\mu_X\sigma_Y^2 - \rho\sigma_X\sigma_Y\mu_Y + (\mu_Y\sigma_X^2 - \rho\sigma_X\sigma_Y\mu_X)w}{(\sigma_Y^2 - 2w\rho\sigma_X\sigma_Y + w^2\sigma_X^2)^{3/2}} .$$

As clearly shown in Frosini (1970), the function $\tilde{F}_W(\cdot)$ is not a proper cdf since it is decreasing on some subset of the support of W. Analogously, $\tilde{f}_W(\cdot)$ is not a well-defined density since it is negative on the subset of the support of W where $F_W(\cdot)$ is decreasing. However, the subset over which $\tilde{F}_W(\cdot)$ ($\tilde{f}_W(\cdot)$) are decreasing (negative) vanishes when $\mu_X/\sigma_X \to \infty$.

A similar approximation can be derived assuming that $\mu_X < 0$ and $|\mu_X| >> \sigma_X$. In this case

$$P\left(\frac{Y}{X} \leq w\right) \approx P(Y - wX \geq 0)$$

and

$$F_W(w) \approx \Phi\left(\frac{-w\mu_X + \mu_Y}{\sqrt{\sigma_Y^2 - 2w\rho\sigma_X\sigma_Y + w^2\sigma_X^2}}\right) . \tag{13}$$

Equations (11) and (13) imply that the random variable

$$\frac{W - \frac{\mu_Y}{\mu_X}}{\sqrt{\frac{\sigma_Y^2}{\mu_X^2} - 2W\frac{\rho\sigma_X\sigma_Y}{\mu_X^2} + W^2\frac{\sigma_X^2}{\mu_X^2}}} \tag{14}$$

approximately follows the standard normal distribution. That result resembles the approximation obtained via the Taylor expansion, up to the first order, of the random variable $\frac{Y}{X}$ around the point $(\mu_Y, \mu_Y) \neq (0, 0)$:

$$\frac{Y}{X} - \frac{\mu_Y}{\mu_X} \approx \frac{1}{\mu_X}(Y - \mu_Y) - \frac{\mu_Y}{\mu_X^2}(X - \mu_X) . \tag{15}$$

As suggested by (15), the random variable

$$\frac{W - \frac{\mu_Y}{\mu_X}}{\sqrt{\frac{\sigma_Y^2}{\mu_X^2} - 2\left(\frac{\mu_Y}{\mu_X}\right)\frac{\rho\sigma_X\sigma_Y}{\mu_X^2} + \left(\frac{\mu_Y}{\mu_X}\right)^2\frac{\sigma_X^2}{\mu_X^2}}} \tag{16}$$

approximately follows the standard normal distribution. Obviously, the normal approximation (16) can be very poor. However, the conditions under which (14) works well suggest that, at least, the standard deviation of X should be small relatively to the absolute value of μ_X. A more precise theoretical framework in which the normal approximation of (16) could be accurate is provided by the well known ''Delta Method'' (see Casella and Berger, 2002, Section 5.5.4). Herebelow, we provide a Lemma which directly follows from its application.

Lemma 1

Let $(Y, X) \sim N\left(\mu_Y, \mu_X, \sigma_Y^2, \sigma_X^2, \rho\right)$ *where* $\mu_X \neq 0$, $\mu_Y \neq 0$, $\sigma_X^2 = \frac{\tilde{\sigma}_X^2}{m}$, $\sigma_Y^2 = \frac{\tilde{\sigma}_Y^2}{m}$, *and* $m > 0$. *The random variable*

$$\sqrt{m}\frac{W - \frac{\mu_Y}{\mu_X}}{\sqrt{\frac{\tilde{\sigma}_Y^2}{\mu_X^2} - 2\left(\frac{\mu_Y}{\mu_X}\right)\frac{\rho\tilde{\sigma}_X\tilde{\sigma}_Y}{\mu_X^2} + \left(\frac{\mu_Y}{\mu_X}\right)^2\frac{\tilde{\sigma}_X^2}{\mu_X^2}}} = \frac{W - \frac{\mu_Y}{\mu_X}}{\sqrt{\frac{\sigma_Y^2}{\mu_X^2} - 2\left(\frac{\mu_Y}{\mu_X}\right)\frac{\rho\sigma_X\sigma_Y}{\mu_X^2} + \left(\frac{\mu_Y}{\mu_X}\right)^2\frac{\sigma_X^2}{\mu_X^2}}}$$

converges in distribution, when $m \to \infty$, *to the standard normal:*

$$\sqrt{m}\frac{W - \frac{\mu_Y}{\mu_X}}{\sqrt{\frac{\tilde{\sigma}_Y^2}{\mu_X^2} - 2\left(\frac{\mu_Y}{\mu_X}\right)\frac{\rho\tilde{\sigma}_X\tilde{\sigma}_Y}{\mu_X^2} + \left(\frac{\mu_Y}{\mu_X}\right)^2\frac{\tilde{\sigma}_X^2}{\mu_X^2}}} \xrightarrow[m\to\infty]{d} N(0,1)\ .$$

It is interesting to note that, under the assumption of Lemma 1 we have:

$$\lim_{m\to\infty} p = \lim_{m\to\infty} e^{\frac{1}{2}(a^2+b^2)} = 0\ .$$

Thanks to (8), it follows that, under the assumption of Lemma 1, the ''Cauchy component'' does not play any role in determining the shape of the limiting distribution of W.

The approximations in Lemma 1 and (14) are linked via Slutsky's theorem (see Casella and Berger, 2002, Theorem 5.5.17 on page 239). Indeed, under the assumption of Lemma 1, it results that

$$W \xrightarrow[m\to\infty]{p} \frac{\mu_Y}{\mu_X}$$

where $\xrightarrow{p}$ denotes convergence in probability. Consequently, the continuous mapping theorem (see Casella and Berger, 2002, Theorem 5.5.4 on page 233) assures that

$$\sqrt{\frac{\sigma_Y^2}{\mu_X^2} - 2W\frac{\rho\sigma_X\sigma_Y}{\mu_X^2} + W^2\frac{\sigma_X^2}{\mu_X^2}} \xrightarrow[m\to\infty]{p} \sqrt{\frac{\sigma_Y^2}{\mu_X^2} - 2\left(\frac{\mu_Y}{\mu_X}\right)\frac{\rho\sigma_X\sigma_Y}{\mu_X^2} + \left(\frac{\mu_Y}{\mu_X}\right)^2\frac{\sigma_X^2}{\mu_X^2}}\ .$$

From Slutsky's theorem it results:

$$\frac{W-\frac{\mu_Y}{\mu_X}}{\sqrt{\frac{\sigma_Y^2}{\mu_X^2}-2W\frac{\rho\sigma_X\sigma_Y}{\mu_X^2}+W^2\frac{\sigma_X^2}{\mu_X^2}}}\xrightarrow[m\to\infty]{d}\frac{W-\frac{\mu_Y}{\mu_X}}{\sqrt{\frac{\sigma_Y^2}{\mu_X^2}-2\left(\frac{\mu_Y}{\mu_X}\right)\frac{\rho\sigma_X\sigma_Y}{\mu_X^2}+\left(\frac{\mu_Y}{\mu_X}\right)^2\frac{\sigma_X^2}{\mu_X^2}}}.$$

4. DISTRIBUTION OF $V = |W|$ AND TRUNCATED DISTRIBUTION OF W

In the following, the distribution of the random variable $V = |W| = \frac{Y}{X}$ will be very useful. It can be easily obtained from (1), (4), (5), and (6) by using relations

$$F_V(v) = F_W(v) - F_W(-v) \quad \textit{with } v \geq 0\,, \tag{17}$$

$$f_V(v) = f_W(v) + f_W(-v) \quad \textit{with } v \geq 0\,.$$

In particular, starting from (5), Pollastri and Tulli (2012, 2015) introduced the formula:

$$\begin{aligned}F_V(v) = {} & \frac{1}{\pi}[\arctan(t_v) - \arctan(t_{-v})] + \\ & + \frac{1}{\pi}\left[\arctan\left(\frac{b+at_v}{bt_v-a}\right) - \arctan\left(\frac{b+at_{-v}}{bt_{-v}-a}\right)\right] + \\ & - 2T\left(\frac{bt_v-a}{\sqrt{1+t_v^2}}, \frac{b+at_v}{bt_v-a}\right) + 2T\left(\frac{bt_{-v}-a}{\sqrt{1+t_{-v}^2}}, \frac{b+at_{-v}}{bt_{-v}-a}\right).\end{aligned}$$

Now, let us consider the random variables W_a^b obtained truncating W in the points a and b with $a < b$. The random variable W_a^b assumes values only in the interval (a, b) and its cdf is given by

$$F_{W_a^b}(w) = \begin{cases} 0 & w \leq a \\ \frac{F_W(w)-F_W(a)}{F_W(b)-F_W(a)} & a < w < b. \\ 1 & w \geq b \end{cases} \tag{18}$$

Analogously, it results that

$$f_{W_a^b}(w) = \begin{cases} 0 & w \leq a \\ \frac{f_W(w)}{F_W(b)-F_W(a)} & a < w < b. \\ 0 & w \geq b \end{cases}$$

5. ASYMPTOTIC CONFIDENCE INTERVALS FOR THE RATIO OF TWO PARAMETERS: DIFFERENT APPROACHES

In this section we will show how to use the distribution of W, V, and W_a^b in the construction of asymptotic confidence intervals for the ratio. Specifically, two cases can be considered:

Case 1: let Z be a random variable whose distribution f_Z depends on two unknown parameters θ_1 and θ_2. A random sample $(Z_1, Z_2, ..., Z_i, ..., Z_n)$ of size n is drawn from f_Z and it is used to define the estimators $\hat{\theta}_1$ and $\hat{\theta}_2$ of θ_1 and θ_2, respectively.

Case 2: let (Z_1, Z_2) be a bivariate random variable with joint density f_Z. Let f_{Z_1} and f_{Z_2} be marginal distributions of Z_1 and Z_2 which depend on two unknown parameters θ_1 and θ_2, respectively. A random sample

$$\{(Z_{11}, Z_{21}), ..., (Z_{1i}, Z_{2i}), ..., (Z_{1n}, Z_{2n})\}$$

of size n is drawn from f_Z and it is used to define the estimators $\hat{\theta}_1$ and $\hat{\theta}_2$.

In both cases it is of interest to find a confidence interval for the ratio $r = \frac{\theta_1}{\theta_2}$ starting from the estimator $R = \frac{\hat{\theta}_1}{\hat{\theta}_2}$. Typically, the multivariate central limit theorem or some other approximation techniques guarantee that

$$\sqrt{n}\begin{bmatrix} \hat{\theta}_1 - \theta_1 \\ \hat{\theta}_2 - \theta_2 \end{bmatrix} \xrightarrow[n\to\infty]{d} \mathbf{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix}\right). \tag{19}$$

In such case, the estimator $R = \frac{\hat{\theta}_1}{\hat{\theta}_2}$ is a ratio of two asymptotically normally distributed random variables. Following the results in Section 3, the exact distribution of R can be approximated by using the Delta Method or by the distribution of the ratio of two CNRV. Intuitively, the distribution of the ratio of two CNRV could provide a better approximation of the exact distribution of R since it involves only the asymptotic approximation (19) and not also the Taylor series approximation (15) implicit in Delta Method adoption. This intuition suggests that a confidence interval for r based on the distribution of the ratio of two CNRV should provide a better coverage accuracy with respect to the one based on the Delta Method. Specifically, let S_1^2, S_2^2, S_{12} be consistent estimators for σ_1^2, σ_2^2, and σ_{12}, respectively. The following confidence intervals for r, with nominal coverage probability $(1-\alpha)$, can be used:

Confidence Interval based on Delta Method (IC_1^α)

$$IC_1^\alpha \equiv \left(R - z_{1-\frac{\alpha}{2}}\sqrt{\frac{\hat{\sigma}_R^2}{n}};\ R + z_{1-\frac{\alpha}{2}}\sqrt{\frac{\hat{\sigma}_R^2}{n}}\right) \tag{20}$$

where

$$\hat{\sigma}_R^2 = \frac{S_1^2}{\hat{\theta}_2^2} - 2R\frac{S_{12}}{\hat{\theta}_2^2} + R^2\frac{S_2^2}{\hat{\theta}_2^2}\ .$$

Confidence Interval based on the ratio of CNRV (IC_2^α, see Galeone, 2007, and Galeone and Pollastri, 2012, 2013)

$$IC_2^{\alpha} \equiv \left(\widehat{F}_R^{-1}(\alpha/2),\, \widehat{F}_R^{-1}(1-\alpha/2)\right) \tag{21}$$

where

$$\widehat{F}_R(w) = L\left(\frac{A - BT_w}{\sqrt{1+T_w^2}},\, -B,\, \frac{T_w}{\sqrt{1+T_w^2}}\right) + L\left(\frac{BT_w - A}{\sqrt{1+T_w^2}},\, B,\, \frac{T_w}{\sqrt{1+T_w^2}}\right)$$

with $w \in R$,

$$\hat{\rho} = \frac{S_{12}}{S_1 S_2}\ , A = \sqrt{\frac{n}{1-\hat{\rho}^2}}\left(\frac{\hat{\theta}_1}{S_1} - \hat{\rho}\frac{\hat{\theta}_2}{S_2}\right),$$

$$B = \sqrt{n}\left(\frac{\hat{\theta}_2}{S_2}\right), \quad T_w = \sqrt{\frac{1}{1-\hat{\rho}^2}}\left(\frac{S_2}{S_1}w - \hat{\rho}\right).$$

Note that (21) asymptotically has a coverage probability of $(1-\alpha)$ since it is asymptotically equivalent to (20) thanks to Lemma 1.

If ''a priori'', it is known that the ratio $r = \frac{\theta_1}{\theta_2}$ is greater than 0, intuition suggests that the exact distribution of $\frac{\hat{\theta}_1}{\hat{\theta}_2}$ can be better approximated by the distribution of the absolute value of the ratio. In this case, a possible improved confidence interval for r results as follows.

Confidence Interval based on the absolute value of the ratio of CNRV (IC_3^{α})

$$IC_3^{\alpha} \equiv \left(\widehat{F}_{|R|}^{-1}(\alpha/2),\, \widehat{F}_{|R|}^{-1}(1-\alpha/2)\right) \tag{22}$$

where, thanks to (17), $\widehat{F}_{|R|}(w) = \widehat{F}_R(w) - \widehat{F}_R(-w)$.

Analogously, if ''a priori'' it is known that the ratio $r = \frac{\theta_1}{\theta_2}$ assumes values only in the interval (a, b), intuition suggests that the exact distribution on $\frac{\hat{\theta}_1}{\hat{\theta}_2}$ can be better approximated by the distribution of R truncated in a and b (see Section 4). In this case, a possible improved confidence interval for r results as follows.

Confidence Interval based on the truncated distribution of the ratio of CNRV (IC_4^{α}(a,b))

$$IC_4^{\alpha}(a,b) \equiv \left(\widehat{F}_{R_a^b}^{-1}(\alpha/2),\, \widehat{F}_{R_a^b}^{-1}(1-\alpha/2)\right) \tag{23}$$

where, thanks to (18), it results

$$\widehat{F}_{R_a^b}(w) = \begin{cases} 0 & w \le a \\ \frac{\widehat{F}_R(w) - \widehat{F}_R(a)}{\widehat{F}_R(b) - \widehat{F}_R(a)} & a < w < b. \\ 1 & w \ge b \end{cases}$$

6. SIMULATION STUDY

In this section, several simulation studies are performed in order to evaluate the eventual gain in coverage accuracy of confidence intervals IC_2^α, IC_3^α, and $IC_4^\alpha(a,b)$ with respect to IC_1^α. Simulations concern 4 different scenarios: a) Confidence interval for the ratio of two means; b) Confidence interval for the Gini Concentration Ratio; c) Confidence interval for the Coefficient of Variation; d) Confidence interval for the Sharpe Ratio. As mentioned in the Introduction, the simulations related to the Sharpe Ratio are performed under a time series setting while the remaining ones are performed assuming simple random sampling.

6.1 *Coverage accuracy of CIs for the ratio of expected values*

Accordingly to the framework described in Case 2 of Section 5, let us assume that the random sample $\{(Y_1,X_1),\dots,(Y_i,X_i),\dots,(Y_n,X_n)\}$ is drawn from (Y,X) in order to find a confidence interval for $\frac{E[Y]}{E[X]}=\frac{\mu_Y}{\mu_X}$. All the settings considered in the simulations satisfy the conditions $\mu_Y>0$ and $\mu_X>0$. Consequently also the confidence intervals IC_3^α and $IC_4^\alpha(0,\infty)$ can be considered as possible competitors in our evaluations. The design of the simulation study is described below:

- **Number of simulations**. For each scenario, 5000 simulations are performed.
- **Sample size**. We considered 6 different sample sizes ranging from 25 to 800:

$$n = 25,\ 50,\ 100,\ 200,\ 400,\ 800\ .$$

- **Nominal Coverage Probability**. We considered the most widespread nominal coverages: $(1-\alpha)=(0.99;\ 0.95;\ 0.9)$.
- **Distribution of** (Y,X). We assume that Y and X follow the beta distribution and the joint distribution of (X,Y) is obtained by using the Gaussian copula. Nine parameters settings are considered. They are detailed in Table 1 where expectations and standard deviations of marginal distributions are provided along with of the natural parameters (h,k) of the beta density:

$$f(x)=\frac{1}{B(h,k)}x^{h-1}(1-x)^{k-1}\quad x\in[0,1],\ h>0,\ k>0\ ,$$

  with $B(h,k)$ the beta function. Note that, in the case of Beta random variables, the parameter ρ can not be interpreted as the correlation coefficient between Y and X but only as measure of concordance between Y and X.

  All the CIs are derived remembering that

$$\sqrt{n}\begin{bmatrix}\bar{Y}-\mu_Y\\ \bar{X}-\mu_X\end{bmatrix}\xrightarrow[n\to\infty]{d}\mathbf{N}\left(\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}\sigma_Y^2 & \sigma_X\sigma_X\rho\\ \sigma_X\sigma_X\rho & \sigma_X^2\end{bmatrix}\right).$$

An estimator for the variance-covariance matrix is necessary in order to define the CIs. We used the classic estimator:

$$\begin{bmatrix} S_Y^2 & S_{XY} \\ S_{XY} & S_Y^2 \end{bmatrix} = \frac{1}{(n-1)} \sum_{i=1}^{n} \begin{bmatrix} (Y_i - \bar{Y})^2 & (Y_i - \bar{Y})(X_i - \bar{X}) \\ (Y_i - \bar{Y})(X_i - \bar{X}) & (X_i - \bar{X})^2 \end{bmatrix}.$$

Simulation results are reported in Tables 2-4. They show that IC_2^α, IC_3^α, and IC_4^α perform very similarly: in the most cases their simulated coverage probabilities are identical and, in the remaining cases, they are almost equal. The coverage probability of IC_1^α is very close to the ones of the other CIs and it is difficult to identify the typical situations in which a confidence interval outperforms the others. At a first look, IC_1^α performs best in almost half cases but it seems that its absolute gain in coverage accuracy when it results the best performer is lower with respect to the loss when it does not result the best performer. Then, globally it seems that IC_2^α (and its modifications) has a slightly better performance. To corroborate this observation we compute, for each CIs, the following indicator:

$$RE(IC_i) = \frac{1}{\#\mathcal{S} \cdot 3 \cdot 6} \sum_{s \in S} \sum_{\alpha} \sum_{n} \frac{|SC(i, \alpha, n, s) - (1-\alpha)|}{(i-\alpha)}$$

where $i = 1, \ldots, 4$, $\mathcal{S}$ is a particular set of distribution settings, $\#\mathcal{S}$ is the number of elements in $\mathcal{S}$, and $SC(i, \alpha, n, s)$ is the simulated coverage probability of IC_i^α at the nominal confidence level $(1-\alpha)$ when the sample size equals n under the simulation setting s. RE can be interpreted as an averaged Relative Error for a certain confidence interval. Computing the RE indicator over all the $(\#\mathcal{S}) \times (\#\alpha) \times (\#n) = 9 \cdot 3 \cdot 6 = 162$ cases, it results

TABLE 1: *Parameters settings considered in the simulations on the CIs for $\frac{\mu_Y}{\mu_X}$ when Y and X follows the Beta distribution with Gaussian Copula.*

Marginal Beta with Gaussian Copula									
Setting #	h_Y	k_Y	h_X	k_X	ρ	μ_Y	μ_X	σ_Y	σ_X
1a	1	3	2	2	0.1	0.25	0.5	$\sqrt{0.0375}$	$\sqrt{0.05}$
1b	1	3	2	2	0.5	0.25	0.5	$\sqrt{0.0375}$	$\sqrt{0.05}$
1c	1	3	2	2	0.9	0.25	0.5	$\sqrt{0.0375}$	$\sqrt{0.05}$
2a	0.5	0.5	1	3	0.1	0.5	0.5	$\sqrt{0.05}$	$\sqrt{0.0375}$
2b	0.5	0.5	1	3	0.5	0.5	0.5	$\sqrt{0.05}$	$\sqrt{0.0375}$
2c	0.5	0.5	1	3	0.9	0.5	0.5	$\sqrt{0.05}$	$\sqrt{0.0375}$
3a	0.5	0.5	2	2	0.1	0.5	0.5	$\sqrt{0.125}$	$\sqrt{0.05}$
3b	0.5	0.5	2	2	0.5	0.5	0.5	$\sqrt{0.125}$	$\sqrt{0.05}$
3c	0.5	0.5	2	2	0.9	0.5	0.5	$\sqrt{0.125}$	$\sqrt{0.05}$

The expectations and the standard deviations of the marginal distributions are given along with the natural parameters of the beta distribution. Note that ρ can not be interpreted as a correlation coefficient even if it measures the strength of the concordance between X and Y

$$RE(IC_1) = 0.0052\ , RE(IC_2) = RE(IC_3) = RE(IC_4) = 0.0051$$

confirming that all the considered CIs have a very similar performance but IC_2^α (and its modifications) are slightly better.

6.2 *Coverage accuracy of CIs for the Coefficient of Variation*

Accordingly to the framework described in Case 1 of Section 5, let us assume that the random sample $(Z_1, Z_2, ..., Z_i, ..., Z_n)$ is drawn from Z in order to find a confidence interval for the Coefficient of Variation $cv = \frac{\sigma_Z}{\mu_Z}$. It is well known that the Coefficient of Variation is meaningful only if $\mu_Z > 0$. Consequently, we considered only simulations settings consistent with this condition. As in the previous case, the confidence intervals IC_3^α and $IC_4^\alpha(0, \infty)$ can be considered as possible competitors in our evaluations. The design of the simulation study is described below:

- **Number of simulations**. For each scenario 5000 simulations are performed.
- **Sample size**. We considered 6 different sample sizes ranging from 25 to 800:

$$n = 25,\ 50,\ 100,\ 200,\ 400,\ 800\ .$$

- **Nominal Coverage Probability**. We considered the most widespread nominal coverages: $(1 - \alpha) = (0.99;\ 0.95;\ 0.9)$.
- **Distribution of** Z. We considered two main cases. In the first one, Z follows the normal distribution. In the second one, Z follows the beta distribution. For each of the two cases, 4 parameters settings are considered. They are:
  Normal distribution: *Setting 1:* $\mu_Z = \sigma_Z = 1$; *Setting 2:* $\mu_Z = \sigma_Z = 5$; *Setting 3:* $\mu_Z = 0.5,\ \sigma_Z = 5$; *Setting 4:* $\mu_Z = 5,\ \sigma_Z = 0.5$.
  Beta distribution: *Setting 1:* $h = k = 0.5$ *which corresponds to* $\mu_Z = 0.5$ *and* $\sigma_Z = \sqrt{0.125}$; *Setting 2:* $h = k = 2$ *which corresponds to* $\mu_Z = 0.5$ *and* $\sigma_Z = \sqrt{0.05}$; *Setting 3:* $h = 1,\ k = 3$ *which corresponds to* $\mu_Z = 0.25$ *and* $\sigma_Z = \sqrt{0.375}$; *Setting 4:* $h = 3,\ k = 1$ *which corresponds to* $\mu_Z = 0.75$ *and* $\sigma_Z = \sqrt{0.375}$.

All the CIs are derived remembering that

$$\sqrt{n}\begin{bmatrix} S - \sigma_Z \\ \bar{Z} - \mu_Z \end{bmatrix} \xrightarrow[n\to\infty]{d} \mathbf{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{\mu_4 - \sigma_Z^2}{4\sigma_Z^2} & \frac{\mu_3}{2\sigma_Z} \\ \frac{\mu_3}{2\sigma_Z} & \sigma_Z^2 \end{bmatrix}\right)$$

where $\mu_3 = E\left[(Z - \mu_Z)^3\right]$ and $\mu_4 = E\left[(Z - \mu_Z)^4\right]$. An estimator of the variance-covariance matrix is necessary in order to define the CIs. We used the classic estimators, which is defined as follows:

TABLE 2: *Simulated coverage probability (in percentage) of CIs for $\frac{\mu_Y}{\mu_X}$ when Y and X follow the Beta distribution with Gaussian Copula with parameters specified in Setting 1a-1c of Table 1*

	Setting 1a				Setting 1b				Setting 1c			
Nominal coverage 90% ($\alpha = 0.1$)												
n	IC_1^α	IC_2^α	IC_3^α	IC_4^α	IC_1^α	IC_2^α	IC_3^α	IC_4^α	IC_1^α	IC_2^α	IC_3^α	IC_4^α
25	87.64	88.08	88.08	88.08	88.50	88.64	88.64	88.64	86.84	87.20	87.20	87.22
50	89.22	89.32	89.32	89.32	89.12	89.30	89.30	89.30	88.28	88.28	88.28	88.26
100	90.32	90.28	90.28	90.28	89.74	89.80	89.80	89.80	88.72	89.00	89.00	88.96
200	89.92	90.04	90.04	90.04	90.18	90.26	90.26	90.26	89.58	89.84	89.84	89.84
400	89.40	89.30	89.30	89.28	89.50	89.58	89.58	89.56	90.66	90.62	90.62	90.58
800	89.82	89.72	89.72	89.72	90.28	90.34	90.34	90.34	89.32	89.18	89.18	89.16
Nominal coverage 95% ($\alpha = 0.05$)												
n	IC_1^α	IC_2^α	IC_3^α	IC_4^α	IC_1^α	IC_2^α	IC_3^α	IC_4^α	IC_1^α	IC_2^α	IC_3^α	IC_4^α
25	93.00	93.44	93.44	93.44	93.60	93.76	93.76	93.76	92.52	92.44	92.44	92.46
50	94.50	94.64	94.64	94.64	94.34	94.48	94.48	94.48	93.84	93.72	93.72	93.70
100	94.80	94.86	94.86	94.84	95.06	95.14	95.14	95.14	94.66	94.90	94.90	94.88
200	95.04	95.06	95.06	95.06	95.50	95.48	95.48	95.50	94.58	94.60	94.60	94.60
400	94.86	94.86	94.86	94.86	95.04	95.12	95.12	95.10	94.94	94.92	94.92	94.96
800	94.98	95.06	95.06	95.06	94.76	94.76	94.76	94.80	94.74	94.88	94.88	94.86
Nominal coverage 99% ($\alpha = 0.01$)												
n	IC_1^α	IC_2^α	IC_3^α	IC_4^α	IC_1^α	IC_2^α	IC_3^α	IC_4^α	IC_1^α	IC_2^α	IC_3^α	IC_4^α
25	97.64	97.88	97.88	97.88	97.60	97.84	97.84	97.84	98.00	97.78	97.78	97.78
50	98.22	98.76	98.76	98.76	98.44	98.56	98.56	98.56	98.52	98.42	98.42	98.42
100	98.88	99.04	99.04	99.04	98.64	98.64	98.64	98.64	98.96	98.86	98.86	98.86
200	98.48	98.62	98.62	98.62	99.00	99.04	99.04	99.04	98.80	98.68	98.68	98.68
400	98.70	98.78	98.78	98.78	98.88	98.88	98.88	98.88	98.90	98.82	98.82	98.82
800	99.00	98.96	98.96	98.98	98.68	98.66	98.66	98.66	98.98	98.94	98.94	98.94

TABLE 3: *Simulated coverage probability (in percentage) of CIs for $\frac{\mu_Y}{\mu_X}$ when Y and X follow the Beta distribution with Gaussian Copula with parameters specified in Setting 2a-2c of Table 1*

	Setting 2a				**Setting 2b**				**Setting 2c**			
Nominal coverage 90% ($\alpha = 0.1$)												
n	IC_1^α	IC_2^α	IC_3^α	IC_4^α	IC_1^α	IC_2^α	IC_3^α	IC_4^α	IC_1^α	IC_2^α	IC_3^α	IC_4^α
25	88.70	88.16	88.16	88.16	89.12	89.00	89.00	89.00	88.34	89.20	89.20	89.18
50	89.74	89.14	89.14	89.14	88.96	89.04	89.04	89.04	89.24	89.68	89.68	89.68
100	89.96	89.32	89.32	89.32	90.00	90.02	90.02	90.02	89.78	89.52	89.52	89.52
200	89.78	89.70	89.70	89.70	89.64	89.70	89.70	89.70	90.02	90.02	90.02	90.02
400	90.02	90.06	90.06	90.06	90.28	90.20	90.20	90.20	90.02	90.04	90.04	90.04
800	89.90	89.90	89.90	89.90	90.04	89.98	89.98	89.98	89.46	89.48	89.48	89.48
Nominal coverage 95% ($\alpha = 0.05$)												
n	IC_1^α	IC_2^α	IC_3^α	IC_4^α	IC_1^α	IC_2^α	IC_3^α	IC_4^α	IC_1^α	IC_2^α	IC_3^α	IC_4^α
25	93.74	94.04	94.04	94.04	93.52	93.78	93.78	93.78	92.86	94.02	94.02	94.02
50	94.78	94.64	94.64	94.64	94.36	94.04	94.04	94.04	93.66	93.80	93.80	93.80
100	95.08	94.74	94.74	94.74	95.02	95.12	95.12	95.12	94.84	95.26	95.26	95.24
200	95.24	95.06	95.06	95.06	94.28	94.12	94.12	94.12	94.80	94.78	94.78	94.78
400	94.96	94.86	94.86	94.86	94.86	94.64	94.64	94.64	94.32	94.28	94.28	94.28
800	94.86	94.82	94.82	94.82	95.20	95.32	95.32	95.32	95.16	95.20	95.20	95.24
Nominal coverage 99% ($\alpha = 0.01$)												
n	IC_1^α	IC_2^α	IC_3^α	IC_4^α	IC_1^α	IC_2^α	IC_3^α	IC_4^α	IC_1^α	IC_2^α	IC_3^α	IC_4^α
25	97.88	97.94	97.94	97.94	98.04	98.18	98.18	98.18	97.66	98.38	98.38	98.38
50	98.52	98.86	98.86	98.86	98.62	98.66	98.66	98.66	98.06	98.58	98.58	98.58
100	98.56	98.60	98.60	98.60	98.66	98.72	98.72	98.72	98.96	98.96	98.96	98.96
200	99.04	99.02	99.02	99.02	98.90	99.00	99.00	99.00	98.90	98.98	98.98	98.98
400	98.84	99.00	99.00	99.00	99.10	99.20	99.20	99.20	98.68	98.90	98.90	98.90
800	98.94	99.04	99.04	99.02	98.92	98.90	98.90	98.90	98.96	99.12	99.12	99.12

TABLE 4: *Simulated coverage probability (in percentage) of CIs for $\frac{\mu_Y}{\mu_X}$ when Y and X follow the Beta distribution with Gaussian Copula with parameters specified in Setting 3a-3c of Table 1*

	Setting 3a				**Setting 3b**				**Setting 3c**			
Nominal coverage 90% ($\alpha = 0.1$)												
n	IC_1^α	IC_2^α	IC_3^α	IC_4^α	IC_1^α	IC_2^α	IC_3^α	IC_4^α	IC_1^α	IC_2^α	IC_3^α	IC_4^α
25	88.92	88.60	88.60	88.60	89.56	89.08	89.08	89.08	89.48	89.08	89.08	89.08
50	89.98	89.56	89.56	89.56	88.94	88.70	88.70	88.68	89.90	89.88	89.88	89.88
100	89.64	89.70	89.70	89.70	90.26	90.12	90.12	90.12	89.34	89.26	89.26	89.26
200	90.52	90.30	90.30	90.30	88.86	88.82	88.82	88.82	89.22	89.28	89.28	89.28
400	90.20	90.16	90.16	90.16	90.60	90.54	90.54	90.54	89.92	89.80	89.80	89.82
800	90.54	90.34	90.34	90.34	89.68	89.68	89.68	89.68	90.06	90.08	90.08	90.04
Nominal coverage 95% ($\alpha = 0.05$)												
n	IC_1^α	IC_2^α	IC_3^α	IC_4^α	IC_1^α	IC_2^α	IC_3^α	IC_4^α	IC_1^α	IC_2^α	IC_3^α	IC_4^α
25	93.84	93.82	93.82	93.82	94.68	94.32	94.32	94.32	94.12	93.96	93.96	93.96
50	94.58	94.50	94.50	94.50	94.22	93.86	93.86	93.86	94.24	94.34	94.34	94.34
100	94.86	94.48	94.48	94.48	94.80	94.72	94.72	94.72	94.64	94.60	94.60	94.60
200	94.78	94.88	94.88	94.88	94.74	94.74	94.74	94.74	94.64	94.60	94.60	94.60
400	95.22	95.30	95.30	95.30	95.00	94.94	94.94	94.94	95.00	94.84	94.84	94.84
800	95.08	95.06	95.06	95.06	94.88	94.92	94.92	94.92	95.26	95.28	95.28	95.28
Nominal coverage 99% ($\alpha = 0.01$)												
n	IC_1^α	IC_2^α	IC_3^α	IC_4^α	IC_1^α	IC_2^α	IC_3^α	IC_4^α	IC_1^α	IC_2^α	IC_3^α	IC_4^α
25	98.30	98.36	98.36	98.36	98.48	98.16	98.16	98.16	98.46	98.32	98.32	98.32
50	98.48	98.26	98.26	98.26	98.90	98.82	98.82	98.82	98.66	98.58	98.58	98.58
100	98.88	98.80	98.80	98.80	98.72	98.64	98.64	98.64	98.70	98.80	98.80	98.80
200	99.08	99.00	99.00	99.00	98.76	98.74	98.74	98.74	99.04	98.98	98.98	98.98
400	98.88	98.82	98.82	98.82	99.06	99.04	99.04	99.04	99.02	99.04	99.04	99.04
800	98.88	98.90	98.90	98.90	99.10	99.06	99.06	99.06	99.06	99.06	99.06	99.06

$$\begin{bmatrix} \frac{\hat{\mu}_4 - S_Z^4}{4S_Z^2} & \frac{\hat{\mu}_3}{2S_Z} \\ \frac{\hat{\mu}_3}{2S_Z} & S_Z^2 \end{bmatrix}$$

where

$$S_Z^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Z_i - \bar{Z})^2 \ , \ \hat{\mu}_3 = \frac{1}{n}\sum_{i=1}^{n}(Z_i - \bar{Z})^3 \ , \ \text{and } \hat{\mu}_4 = \frac{1}{n}\sum_{i=1}^{n}(Z_i - \bar{Z})^4 \ .$$

It is well known that, when Z follows the Normal distribution, the estimator of cv given by $CV = \frac{S}{\bar{Z}}$ is linked to the noncentral t distribution. Specifically, it results that

$$\sqrt{n}\frac{1}{CV} = \sqrt{n}\frac{\bar{Z}}{S} \sim \mathcal{T}_{n-1}\left(\sqrt{n}\frac{\mu}{\sigma}\right)$$

where $\mathcal{T}_l(h)$ denotes the noncentral t distribution with l degrees of freedom and noncentrality parameter h. That result allows to introduce an exact confidence interval for cv by using the method of ''Pivoting a continuous cdf'' (see Casella and Berger, 2002, Theorem 9.2.12 on page 432). We include this interval in the simulation study related to Normal distribution as a benchmark, in order to emphasize the magnitude of the Monte-Carlo variability. In Tables 5-8 the simulated coverage probabilities of the various confidence intervals are given. Concerning the case $Z \sim N(\mu_X, \sigma_X)$ (see Tables 5 and 6) it results that IC_3^α and IC_4^α have a coverage probability which is almost identical to the one of IC_2^α. Comparing the coverage probabilities of IC_1^α and IC_2^α we observe a similar pattern under Settings 1, 2 and 4: both IC_1^α and IC_2^α show undercoverage but the actual coverage of IC_2^α is generally closer to the nominal one with respect to that of IC_1^α. This fact is particularly evident under Setting 1 when $\alpha = 0.05$ or $\alpha = 0.01$ and the sample size is small. In this particular case the gain in coverage accuracy can reach 3% (see the case $\alpha = 0.01$ and $n = 25$ under Setting 1). When $\alpha = 0.1$ the two CIs are substantially equivalent, with only a little advantage for IC_1^α. Not surprisingly, results obtained under Setting 3 are very different. This is not unexpected for IC_1^α since, in this scenario, the expected value of the denominator is very small with respect to its standard deviation. This is in contrast with the assumptions (see Section 4) assuring that the asymptotic Normal approximation works well. Indeed, IC_1^α has a coverage probability which is dramatically smaller than the nominal one, unless the sample size reaches the value 200 or 400. The coverage probability of IC_2^α shows a very different behavior: it is in line with the nominal value when the sample size is very small ($n = 25$ or $n = 50$) but it rapidly increases with the sample size, far exceeding the nominal value. Only when $n = 800$ the increase of the coverage probability stops and the convergence to the nominal value restarts. However, both IC_1^α and IC_2^α are not accurate in this context. Concerning IC_3^α and IC_4^α, their behavior is very similar but IC_3^α is the best performer: it has the correct coverage when the sample size is small (as IC_2^α) and it does not hugely exceed the nominal value when the sample size increases. To corroborate the previous observations we compute, for each CI, the RE indicator. Focusing only on the simulation Settings 1,2, and 4 in Tables 5 and 6, it results that $RE(IC_1) = 0.0198$, $RE(IC_2) = 0.0160$, $RE(IC_3) = 0.0164$, $RE(IC_4) = 0.0164$ confirming that, globally, IC_2 is the best performer under these settings. Under Setting 3 in Table 6, it results that $RE(IC_1) = 0.1358$, $RE(IC_2) = 0.0205$, $RE(IC_3) = 0.0145$, $RE(IC_4) = 0.0185$ confirming the evident supremacy of IC_3.

As regards the cases in which $Z \sim Beta(h, k)$, it results that, in the most cases, IC_2^α,IC_3^α, and IC_4^α have a very similar performance. Significant differences among these CIs can be observed only under Setting 1 in Table 7. IC_1^α is the best performer in half cases. Its good performance is observed especially when the sample size is small. A global evaluation, performed through the index RE previously introduced, gives:

$RE(IC_1) = 0.0502$, $RE(IC_2) = 0.0509$, $RE(IC_3) = 0.0507$, $RE(IC_4) = 0.0504$.

That result confirms that the 4 methodologies are substantially equivalent in this context.

TABLE 5: - *Simulated coverage probability (in percentage) of CIs for cv when $Z \sim N(\mu_Z, \sigma_Z)$ under settings 1 and 2. The columns corresponding to IC_{ex}^α are related to the exact confidence interval for cv based on the noncentral t distribution*

	Setting 1: $\mu_X = \sigma_X = 1$					**Setting 2:** $\mu_X = \sigma_X = 5$				
Nominal coverage 90% ($\alpha = 0.1$)										
n	IC_1^α	IC_2^α	IC_3^α	IC_4^α	IC_{ex}^α	IC_1^α	IC_2^α	IC_3^α	IC_4^α	IC_{ex}^α
25	86.26	86.42	86.20	86.20	89.42	86.80	87.08	86.90	86.90	90.84
50	89.56	88.38	88.38	88.38	89.98	88.00	87.06	87.06	87.06	90.46
100	88.46	88.14	88.14	88.14	90.50	89.26	88.90	88.90	88.90	90.22
200	90.14	90.02	90.02	90.02	89.76	89.44	89.30	89.30	89.30	90.04
400	89.08	89.42	89.42	89.42	89.90	90.16	90.02	90.02	90.02	89.52
800	89.64	89.68	89.68	89.68	90.00	90.50	90.40	90.40	90.40	90.06
Nominal coverage 95% ($\alpha = 0.05$)										
n	IC_1^α	IC_2^α	IC_3^α	IC_4^α	IC_{ex}^α	IC_1^α	IC_2^α	IC_3^α	IC_4^α	IC_{ex}^α
25	89.26	91.44	90.84	90.84	95.44	89.88	91.94	91.38	91.38	95.00
50	92.26	93.44	93.44	93.44	95.24	91.76	93.18	93.18	93.18	95.22
100	93.18	93.82	93.82	93.82	95.44	93.90	94.10	94.10	94.10	94.70
200	94.40	94.78	94.78	94.78	94.72	94.28	94.28	94.28	94.28	94.56
400	94.44	94.64	94.64	94.64	94.90	94.84	95.02	95.02	95.02	94.82
800	94.84	94.96	94.96	94.96	95.02	94.78	94.62	94.62	94.64	95.24
Nominal coverage 99% ($\alpha = 0.01$)										
n	IC_1^α	IC_2^α	IC_3^α	IC_4^α	IC_{ex}^α	IC_1^α	IC_2^α	IC_3^α	IC_4^α	IC_{ex}^α
25	94.02	97.16	96.66	96.66	98.26	93.66	96.80	96.50	96.50	98.24
50	95.90	97.60	97.60	97.60	98.96	96.36	97.96	97.96	97.96	99.00
100	96.92	98.12	98.12	98.12	98.98	97.78	98.46	98.46	98.46	98.98
200	98.62	98.88	98.88	98.88	98.98	98.12	98.74	98.74	98.74	98.98
400	98.46	99.04	99.04	99.04	99.10	98.68	98.90	98.90	98.90	99.00
800	98.72	98.78	98.78	98.78	98.94	98.72	98.88	98.88	98.88	99.26

6.3 *Coverage accuracy of CIs for the Gini Concentration Ratio*

Gini Concentration Ratio (see Zenga, 1998) is a widespread inequality measure. For a given non negative random variable Z, with distribution function $F(\cdot)$ and finite expectation μ_Z, the Gini Concentration Ratio is given by

$$G = \frac{\Delta_Z}{2\mu_Z}$$

where Δ_Z is the Gini mean difference of Z: $\Delta_Z = E[|Z - \tilde{Z}|]$, $\tilde{Z}$ i.i.d. to Z. Accordingly to the framework described in Case 1 of Section 5, let us assume that the random sample $(Z_1, Z_2, ..., Z_i, ..., Z_n)$ is drawn from Z in order to build the confidence intervals for G introduced in Section 4. These confidence intervals can be introduced assuming that $E[Z^2]$ is finite thanks to the well known results on U-statistics due to Hoeffding (1948):

TABLE 6. - *Simulated coverage probability (in percentage) of CIs for cv when $Z \sim N(\mu_Z, \sigma_Z)$ under settings 3 and 4. The columns corresponding to IC^{α}_{ex} are related to the exact confidence interval for cv based on the noncentral t distribution*

	Setting 3: $\mu_X = 0.5; \sigma_X = 5$					**Setting 4:** $\mu_X = 5; \sigma_X = 0.5$				
Nominal coverage 90% $(\alpha = 0.1)$										
n	IC^{α}_1	IC^{α}_2	IC^{α}_3	IC^{α}_4	IC^{α}_{ex}	IC^{α}_1	IC^{α}_2	IC^{α}_3	IC^{α}_4	IC^{α}_{ex}
25	62.84	88.22	91.36	93.06	90.28	81.04	81.00	81.00	81.06	90.58
50	71.22	91.24	92.76	93.80	90.15	85.68	85.78	85.78	85.76	90.02
100	76.82	93.22	93.78	94.36	90.44	88.44	88.54	88.54	88.50	90.60
200	82.64	94.68	94.08	94.86	90.15	88.80	88.82	88.82	88.80	90.64
400	85.46	94.78	92.34	92.58	89.89	89.78	89.66	89.66	89.74	90.26
800	87.88	95.30	90.52	90.02	90.15	89.84	89.84	89.84	89.90	90.38
Nominal coverage 95% $(\alpha = 0.05)$										
n	IC^{α}_1	IC^{α}_2	IC^{α}_3	IC^{α}_4	IC^{α}_{ex}	IC^{α}_1	IC^{α}_2	IC^{α}_3	IC^{α}_4	IC^{α}_{ex}
25	66.78	93.68	95.42	95.86	95.37	87.34	87.30	87.30	87.30	95.66
50	73.62	95.74	96.26	96.88	95.20	90.56	90.64	90.64	90.62	95.14
100	81.14	96.98	97.10	97.34	95.09	93.16	93.20	93.20	93.16	95.14
200	86.00	96.84	96.78	96.94	94.79	94.04	94.04	94.04	94.08	95.08
400	88.60	97.56	96.78	97.56	94.99	94.00	93.98	93.98	94.06	95.26
800	91.04	97.18	95.12	94.54	95.09	94.80	94.86	94.86	94.86	95.02
Nominal coverage 99% $(\alpha = 0.01)$										
n	IC^{α}_1	IC^{α}_2	IC^{α}_3	IC^{α}_4	IC^{α}_{ex}	IC^{α}_1	IC^{α}_2	IC^{α}_3	IC^{α}_4	IC^{α}_{ex}
25	72.94	98.48	98.76	98.84	98.57	93.88	94.02	94.02	94.02	99.22
50	80.82	99.14	99.18	99.28	99.02	96.70	96.74	96.74	96.72	99.10
100	86.46	99.24	99.26	99.32	99.07	97.56	97.64	97.64	97.62	99.24
200	90.72	99.40	99.40	99.42	99.09	98.66	98.68	98.68	98.68	99.30
400	93.02	99.70	99.64	99.70	98.95	98.38	98.36	98.36	98.34	98.74
800	94.86	99.30	99.00	99.12	99.06	98.66	98.66	98.66	98.66	98.98

$$\sqrt{n}\begin{bmatrix}\hat{\Delta}_Z-\Delta_Z\\2\bar{Z}-2\mu_Z\end{bmatrix}\xrightarrow[n\to\infty]{d}\mathbf{N}\left(\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}\zeta_Z^2 & 2\gamma_Z\\2\gamma_Z & 4\sigma_Z^2\end{bmatrix}\right) \tag{24}$$

where

$$\widehat{G}=\frac{\hat{\Delta}_Z}{2\bar{Z}} \qquad \text{and} \qquad \hat{\Delta}_Z=\frac{1}{n(n-1)}\sum_{i=1}^{n}\sum_{j=1}^{n}|Z_i-Z_j|$$

are the natural estimators for G and Δ_Z, respectively. Moreover, $\gamma_Z=2(\mathcal{D}-\mu\Delta)$ and $\zeta_Z^2=4(\mathcal{F}-\Delta^2)$ with

$$\mathcal{D}=\int_0^{\infty}\int_0^{\infty}x|x-y|dF(x)dF(y) \text{ and}$$

TABLE 7. - *Simulated coverage probability (in percentage) of CIs for cv when $Z\sim Beta(a,b)$ under settings 1 and 2*

	Setting 1: $h=0.5;k=0.5$				**Setting 2:** $h=2;k=2$			
Nominal coverage 90% $(\alpha=0.1)$								
n	IC_1^α	IC_2^α	IC_3^α	IC_4^α	IC_1^α	IC_2^α	IC_3^α	IC_4^α
25	89.60	87.54	86.48	87.16	85.68	84.92	84.92	84.90
50	89.58	88.32	88.76	89.06	88.06	87.80	87.80	87.78
100	89.98	89.42	88.60	89.36	88.94	89.06	89.06	89.06
200	89.04	89.20	89.90	89.72	89.50	89.72	89.72	89.68
400	89.94	89.56	88.96	89.60	90.06	89.64	89.64	89.64
800	89.24	89.10	90.56	89.82	89.30	89.40	89.40	89.40
Nominal coverage 95% $(\alpha=0.05)$								
n	IC_1^α	IC_2^α	IC_3^α	IC_4^α	IC_1^α	IC_2^α	IC_3^α	IC_4^α
25	94.06	92.64	92.28	92.08	91.90	91.34	91.34	91.34
50	94.56	93.38	93.28	93.26	93.52	93.14	93.14	93.14
100	94.84	94.82	94.20	94.68	94.74	94.62	94.62	94.62
200	95.48	94.98	94.78	94.98	94.42	94.28	94.28	94.28
400	95.40	95.36	94.64	94.88	94.54	94.32	94.32	94.32
800	94.98	95.18	94.90	94.66	95.20	95.04	95.04	95.08
Nominal coverage 99% $(\alpha=0.01)$								
n	IC_1^α	IC_2^α	IC_3^α	IC_4^α	IC_1^α	IC_2^α	IC_3^α	IC_4^α
25	97.98	97.10	97.40	97.24	96.58	96.84	96.84	96.84
50	98.68	98.00	98.08	97.98	98.00	98.16	98.16	98.16
100	98.78	98.58	98.60	98.54	98.50	98.68	98.68	98.68
200	98.72	98.78	98.96	98.92	98.60	98.72	98.72	98.72
400	98.98	99.08	98.90	98.92	99.14	99.02	99.02	99.02
800	99.08	99.06	98.90	99.08	98.84	98.92	98.92	98.92

TABLE 8. - *Simulated coverage probability (in percentage) of CIs for cv when $Z \sim Beta(a, b)$ under settings 3 and 4*

	Setting 3: $h = 1; k = 3$				**Setting 4:** $h = 3; k = 1$			
Nominal coverage 90% $(\alpha = 0.1)$								
n	IC_1^α	IC_2^α	IC_3^α	IC_4^α	IC_1^α	IC_2^α	IC_3^α	IC_4^α
25	88.02	87.48	87.50	87.50	81.26	80.66	80.66	80.68
50	88.62	89.32	89.32	89.32	85.62	85.58	85.58	85.58
100	89.68	89.32	89.32	89.32	88.12	87.98	87.98	87.98
200	90.12	90.06	90.06	90.06	89.82	89.74	89.74	89.74
400	89.76	89.74	89.74	89.72	89.62	89.66	89.66	89.64
800	89.92	90.08	90.08	90.10	89.92	89.86	89.86	89.86
Nominal coverage 95% $(\alpha = 0.05)$								
n	IC_1^α	IC_2^α	IC_3^α	IC_4^α	IC_1^α	IC_2^α	IC_3^α	IC_4^α
25	92.66	93.32	93.32	93.32	86.38	86.30	86.30	86.30
50	93.50	93.76	93.76	93.76	91.02	91.08	91.08	91.08
100	94.28	94.36	94.36	94.36	92.92	92.80	92.80	92.80
200	94.54	94.44	94.44	94.44	93.96	93.94	93.94	93.94
400	94.04	94.14	94.14	94.10	94.54	94.70	94.70	94.70
800	95.66	95.68	95.68	95.66	94.78	94.78	94.78	94.80
Nominal coverage 99% $(\alpha = 0.01)$								
n	IC_1^α	IC_2^α	IC_3^α	IC_4^α	IC_1^α	IC_2^α	IC_3^α	IC_4^α
25	97.66	98.12	98.12	98.12	92.20	92.74	92.74	92.74
50	98.34	98.78	98.78	98.78	95.76	96.06	96.06	96.06
100	98.82	99.12	99.12	99.12	97.54	97.72	97.72	97.72
200	98.82	98.94	98.94	98.94	98.36	98.44	98.44	98.44
400	98.80	98.88	98.88	98.88	98.84	98.90	98.90	98.88
800	98.94	99.06	99.06	99.06	98.92	99.00	99.00	99.00

$$\mathcal{F} = \int_0^\infty \int_0^\infty \int_0^\infty |x-y||x-z| dF(x) dF(y) dF(z) \; .$$

An unbiased and consistent estimator for ζ_Z^2 was introduced in Zenga (2004):

$$\hat{\zeta}^2 = \frac{4n}{(n-2)(n-3)} \left[S_Z^2 + (n-2)\hat{\mathcal{F}} - \frac{2n-3}{2} \hat{\Delta}_Z^2 \right]$$

where

$$\hat{\mathcal{F}} = \frac{1}{n(n-1)(n-2)} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{l=1}^{n} |X_i - X_j||X_i - X_l| - \frac{2S_Z^2}{n-2} \; .$$

An unbiased and consistent estimator for γ was introduced in Polisicchio (1997):

$$\hat{\gamma} = \frac{2n}{n-2} \left[\hat{\mathcal{D}} - \bar{Z}\hat{\Delta}_Z \right] \quad \text{where} \quad \hat{\mathcal{D}} = \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j=1}^{n} X_i |X_i - X_j| \; .$$

The estimators just recalled allow to estimate all the elements of the variance-covariance matrix in (24) and, then, all the confidence intervals in Section 4 can be introduced. Since $0 \leq G \leq 1$, the interval $IC_3^\alpha(0, 1)$ is considered. The design of the simulation study is the following:

- **Number of simulations.** For each scenario 5000 simulations are performed.
- **Sample size.** We considered 6 different sample sizes ranging from 25 to 800:

$$n = 25,\ 50,\ 100,\ 200,\ 400,\ 800\ .$$

- **Nominal Coverage Probability.** We considered the most widespread nominal coverages: $(1 - \alpha) = (0.99; 0.95; 0.9)$.
- **Distribution of** Z**.** We considered three different distributions: Zenga Distribution, Dagum Distribution, LogNormal Distribution. LogNormal and Dagum distributions are well known and widely used to represent the income distribution (see Chotikapanich, 2008). Zenga Distribution was recently introduced in Zenga (2010) and deeply studied in Zenga, Pasquazzi, Polisicchio, and Zenga (2011), Zenga, Pasquazzi, and Zenga (2012), Arcagni and Porro (2013, 2016), Arcagni and Zenga (2013), De Capitani and Zini (2013a,2013b), Arcagni (2014), Porro (2015). All these studies showed that this distribution describes incomes very well. Only one parameter setting for each distribution has been chosen:
  - *Zenga Distribution.* Following the parametrization in Zenga (2010) we set: $\alpha = 3$, $\theta = 1$, $\mu = 1$. μ is a scale parameter and does not influence the value of G which, in this case, is equal to 0.3659.
  - *Dagum Distribution.* Following the parametrization in Greselin and Pasquazzi (2008) we set: $\theta = 2.05$, $\beta = 1.5$, $\lambda = 1$. λ is a scale parameter and does not influence the value of G which, in this case, is equal to 0.4726. With this parameters setting, the Dagum Distribution has a very fat right tail (its variance exists only if $\theta > 2$).
  - *LogNormal Distribution.* Following the usual parametrization (see, e.g., Mood, Graybill and Boes, 1974) we set: $\sigma = 0.55$, $\mu = 9$. In this case $G = 0.3026$.

Simulations (see Table 9) show that IC_2^α, IC_3^α, and IC_4^α have almost the same coverage accuracy. Compared to IC_1^α, they have a better coverage accuracy when the sample size is small and data are drawn from the Zenga and LogNormal distributions.

When data comes from the Dagum Distribution the improvement is observed when $n = 25$ for all the nominal coverages. In the remaining cases, IC_1^α is the best performer. However, it is worth noting that the global performance of the 4 different CIs is very similar, as confirmed by the RE indexes: $RE(IC_1) = 0.0679$, $RE(IC_2) = 0.0652$, $RE(IC_3) = 0.0652$, $RE(IC_4) = 0.0653$.

TABLE 9. - *Simulated coverage probability (in percentage) of CIs for G when sampling from the Zenga, Dagum and LogNormal distributions*

	Zenga Distribution				Dagum Distribution				LogNormal Distribution			
Nominal coverage 90% ($\alpha = 0.1$)												
n	IC_1^α	IC_2^α	IC_3^α	IC_4^α	IC_1^α	IC_2^α	IC_3^α	IC_4^α	IC_1^α	IC_2^α	IC_3^α	IC_4^α
25	82.0	84.8	84.8	84.8	64.2	67.4	67.4	67.5	84.3	86.1	86.1	86.1
50	82.9	85.3	85.3	85.3	70.4	71.6	71.4	71.3	84.0	85.4	85.4	85.4
100	86.2	86.6	86.6	86.6	75.0	74.3	74.4	74.3	87.0	88.4	88.4	88.4
200	87.0	87.4	87.4	87.4	76.1	76.0	76.0	76.0	87.9	87.8	87.8	87.8
400	88.3	88.4	88.4	88.4	78.9	79.0	79.0	79.0	90.5	90.5	90.5	90.5
800	87.3	87.7	87.7	87.7	77.8	77.6	77.6	77.6	89.4	89.3	89.4	89.3
Nominal coverage 95% ($\alpha = 0.05$)												
n	IC_1^α	IC_2^α	IC_3^α	IC_4^α	IC_1^α	IC_2^α	IC_3^α	IC_4^α	IC_1^α	IC_2^α	IC_3^α	IC_4^α
25	88.2	90.3	90.3	90.3	69.9	71.4	71.3	71.3	90.9	91.6	91.6	91.6
50	89.3	89.9	89.9	89.9	78.7	78.2	78.2	78.2	91.9	92.5	92.5	92.5
100	91.6	91.9	91.9	91.8	82.5	80.2	80.2	80.2	94.2	94.5	94.5	94.5
200	92.5	92.4	92.4	92.4	82.6	81.9	81.9	81.9	93.2	93.2	93.2	93.2
400	92.6	93.0	93.0	93.0	84.0	83.7	83.7	83.7	93.7	93.5	93.6	93.5
800	94.7	94.9	94.9	94.9	87.2	86.0	86.0	86.0	94.1	94.0	94.0	94.0
Nominal coverage 99% ($\alpha = 0.01$)												
n	IC_1^α	IC_2^α	IC_3^α	IC_4^α	IC_1^α	IC_2^α	IC_3^α	IC_4^α	IC_1^α	IC_2^α	IC_3^α	IC_4^α
25	93.9	94.8	94.8	94.8	83.1	84.5	84.5	84.5	95.8	96.8	96.8	96.8
50	96.0	96.8	96.8	96.8	85.1	84.4	84.3	84.4	97.5	97.9	97.9	97.9
100	96.1	96.2	96.2	96.2	88.6	86.7	86.7	86.7	98.2	98.3	98.3	98.3
200	97.6	97.4	97.4	97.4	89.2	87.5	87.5	87.5	98.0	98.0	98.0	98.0
400	97.9	97.9	97.9	97.9	91.7	90.2	90.2	90.2	98.8	98.9	98.9	98.9
800	98.4	98.3	98.3	98.3	93.5	92.2	92.2	92.2	98.6	98.6	98.6	98.6

6.4 *Coverage accuracy of CIs for the Sharpe Ratio*

The Sharpe Ratio (Sharpe 1964, 1966, 1994) is commonly used to measure the risk-adjusted performance of a financial asset and to compare different portfolios of financial activities. It is defined using the standard deviation of returns (interpreted as a risk measure) and the expected excess return (interpreted as reward measure) to determine the reward per unit of risk. Specifically, let Z be the random variable describing the log-return of a risky financial activity and let ξ be the (log) *risk-free rate*. Let μ_Z and σ_Z be the expected value and the standard deviation of Z, respectively. The Sharpe Ratio is given by:

$$\psi = \frac{\mu - \xi}{\sigma} \quad . \tag{25}$$

Let $Z_1, ..., Z_n$ be a time series of returns and let $\bar{Z}$ and S_Z^2 be the sample mean and the unbiased sample variance, respectively. ψ can be estimated by the natural estimator $\widehat{\Psi} = \frac{\bar{X}-\xi}{S}$ which is the ratio of two random variables. Confidence intervals and hypothesis testing on ψ based on the estimator $\widehat{\Psi}$ have been studied in Jobson and Korkie (1981), Lo (2002), Miller and Gehr (1978), Opdyke (2007), De Capitani and Zenga (2011), De Capitani (2012, 2014) and De Capitani and Pasquazzi (2015). Here, we recall the result in De Capitani (2012) which shows that, under certain regularity conditions regarding the strength of temporal dependence, and assuming that the returns process $\{Z_t\}_{t\in\mathbb{N}}$ is strictly stationary with $E[Z_t] < \infty$, it results:

$$\sqrt{n}\begin{bmatrix} (\bar{Z} - \xi) - (\mu_Z - \xi) \\ S_Z - \sigma_Z \end{bmatrix} \xrightarrow[n\to\infty]{d} \mathbf{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12} & \Sigma_{22} \end{bmatrix}\right)$$

where

$$\Sigma_{11} = \sigma_Z^2 + 2\sum_{i=1}^{\infty} \sigma_i \ , \qquad \Sigma_{12} = \frac{\mu_3 + \sum_{i=1}^{\infty}(\mu_i^{2,1} + \mu_i^{1,2})}{2\sqrt{\Sigma_{11}}} \ ,$$

$$\Sigma_{22} = \frac{(\mu_4 - \sigma_Z^2) + 2\sum_{i=1}^{\infty}(\mu_i^{2,2} - \sigma_Z^4)}{4\Sigma_{11}} \ ,$$

with $\mu_h = E[(Z_t - \mu_Z)^h]$, $\sigma_i = E[(Z_{i+1} - \mu_Z)(Z_1 - \mu_Z)]$, and $\mu_i^{h,k} = E[(Z_{i+1} - \mu_Z)^h (Z_1 - \mu_Z)^k]$.

Following De Capitani (2014), the elements of the variance-covariance matrix can be consistently estimated by using the Newey-West estimator (Newey and West, 1987) with the automatic and data-dependent procedure for bandwidth selection proposed in Newey and West (1994) (see Hall, 2005, on page 82-83, for a description of this method). By using this estimator the CIs introduced in Section 4 can be defined. However, in this scenario only IC_1^α (which is the one considered in De Capitani, 2014, 2012) and IC_2^α are considered since ψ is not necessarily non-negative. The design of the simulation study is the following:

- **Number of simulations.** For each scenario 5000 simulations are performed.
- **Sample size.** We considered 6 different sample sizes ranging from 50 to 1600:

$$n = 50,\ 100,\ 200,\ 400,\ 800,\ 1600\ .$$

- **Nominal Coverage Probability.** We considered the most widespread nominal coverages: $(1-\alpha) = (0.99;\ 0.95;\ 0.9)$.
- **Process** $\{Z\}_{t\in N}$. We choose the GARCH(1,1) process with Gaussian innovations:

$$\begin{aligned} Z_t - \mu_Z &= \sigma_t \epsilon_t \\ \sigma_t^2 &= \alpha_0 + \alpha_1 (Z_{t-1} - \mu)^2 + \beta \sigma_{t-1}^2\ . \end{aligned}$$

  where $\alpha_0 > 0$, $\alpha_1 \geq 0$, $\beta \geq 0$ and the random variables ϵ_t, $(t = 1, 2, ...)$, referred to as the innovations of the process, are i.i.d and follow the standard Normal distribution. In the simulations we set $\alpha_0 = 0.001$, $\alpha_1 = 0.1$, $\beta = 0.8$ and $\zeta = 0.000068$. Three values for μ_Z are considered: 0.005068,0.025068, 0.050068. In all these three parameters settings it results that $\sigma_Z^2 = Var(Z_t) = \frac{\alpha_0}{1-(\alpha_1+\beta)} = 0.01$. Thus, the true Sharpe Ratio is equal to 0.05, 0.025, and 0.5 when μ_Z is equal to 0.005068,0.025068, and 0.050068, respectively.

Simulation results are given in Table 10. In 45 out of 54 cases, IC_2^α shows a better coverage accuracy with respect to IC_1^α. The improvement is very small but, under the considered scenario, IC_2^α is undoubtedly the best CI. In order to evaluate the magnitude of the improvement, we computed the RE indicators obtaining the following values:

$$RE(IC_1) = 0.0520, \quad RE(IC_2) = 0.0516\ .$$

7. CONCLUSIONS

In this paper, four different approaches for the definition of asymptotic confidence intervals for the ratio of two unknown parameters are reviewed and compared. The first considered approach is based on the well known Delta Method and stems from the asymptotic normality of the ratio of two asymptotically jointly normal random variables. The remaining approaches are all based on the distribution of the ratio of correlated normal random variables. Roughly speaking, these methods avoid the linearization of the ratio random variable implicit in the Delta Method and, for this reason, intuition suggests that they could provide a better coverage accuracy. To provide some evidences of this fact, a simulation study is performed. Simulations concern the ratio between two expectations, the Coefficient of Variation, the Gini Concentration Ratio, and the Sharpe Ratio. Several nominal coverages, sample sizes, and distributions are considered in the simulations. They show that the asymptotic confidence intervals based on the ratio of correlated normal random variables often have a better coverage accuracy with respect to the ones derived from Delta Method, even if the observed gain is in some cases small.

TABLE 10 - *Simulated coverage probability (in percentage) of CIs for ψ when return's process is GARCH(1,1) with Gaussian innovations*

	$\psi = 0.05$		$\psi = 0.25$		$\psi = 0.5$	
Nominal coverage 90% $(\alpha = 0.1)$						
n	IC_1^α	IC_2^α	IC_1^α	IC_2^α	IC_1^α	IC_2^α
50	85.78	85.61	83.32	83.54	79.84	80.78
100	87.24	87.44	87.04	87.22	83.30	83.78
200	88.84	89.02	86.72	87.04	84.54	85.10
400	88.51	88.66	88.48	88.62	85.68	86.04
800	89.53	89.63	87.38	87.64	86.06	86.32
1600	90.58	90.64	88.34	88.54	87.23	87.36
Nominal coverage 95% $(\alpha = 0.05)$						
n	IC_1^α	IC_2^α	IC_1^α	IC_2^α	IC_1^α	IC_2^α
50	90.64	90.52	90.28	90.46	86.56	86.90
100	93.18	93.46	92.02	92.32	89.84	89.76
200	94.26	94.44	93.93	93.86	90.78	91.01
400	94.84	94.96	94.04	94.38	91.24	91.72
800	94.74	94.80	94.06	94.21	92.12	92.46
1600	95.21	95.24	94.65	94.70	92.26	92.44
Nominal coverage 99% $(\alpha = 0.01)$						
n	IC_1^α	IC_2^α	IC_1^α	IC_2^α	IC_1^α	IC_2^α
50	96.48	96.40	95.96	96.02	93.98	94.31
100	98.22	98.28	97.52	97.48	96.76	96.86
200	98.66	98.71	98.13	98.35	96.88	96.86
400	98.66	98.74	98.20	98.22	97.42	97.58
800	98.92	98.93	98.54	98.72	98.36	98.24
1600	98.76	98.84	98.56	98.56	97.71	97.72

APPENDIX

COMPUTATIONAL DETAILS

In order to compute the distribution function and the quantiles of W by using expression (1), the R Software (R Core Team, 2017) is used. Specifically, the package mvtnorm (Genz *et al.* 2017) is very useful. Indeed, this package contains the function pmvnorm which computes the bivariate normal integral according to the indication of Kotz et al. (2004). The R-function F_W based on expression (1) is given below. The name of its arguments are consistent with the notation in formulas (1), (2), and (3).

```
F_W<-function(w,mu_X,mu_Y,sigma_X,sigma_Y,rho){
   #definition of "a", "b" and "t_w" according to (2) and (3)
   a  <-  (1/(1-rho^2))^0.5*(mu_Y/sigma_Y-rho*mu_X/sigma_X)
   b  <-  mu_X/sigma_X
   t_w <-  (1/(1-rho^2))^0.5*(sigma_X/sigma_Y*w-rho)
   #definition of the bivariate normal integral
   A  <-  (a-b*t_w)/((1+t_w^2)^0.5)
   B  <-  -b
   C  <-  t_w/((1+t_w^2)^0.5)
   #definition of the cdf of Y/X
   Mcor<-matrix(c(1,C,C,1),2,2)
   A1<-pmvnorm(lower=c(A,B),upper=Inf, corr=Mcor)
   A2<-pmvnorm(lower=c(-A,-B),upper=Inf, corr=Mcor)
   res<-A1+A2
   return(res)}
```

Quantiles from $F_W(\cdot)$ can be computed by using the following R-function, where the argument q stands for the quantile level.

```
   Quantile_W<-function(q,mu_X,mu_Y,sigma_X,sigma_Y,rho){
   Q<-uniroot(function(w) F_W(w,mu_X,mu_Y,sigma_X,sigma_Y,rho)-q,
   lower = -1000, upper = 1000, maxiter = 1000,extendInt = "yes")
   return(Q$root)}
```

Analogously, starting from expression (17), quantiles from $F_V(\cdot)$ can be computed by using the following R-function.

```
   Quantile_V<-function(q,mu_X,mu_Y,sigma_X,sigma_Y,rho){
   Q<-uniroot(function(v) (F_W(v,mu_X,mu_Y,sigma_X,sigma_Y,rho)+
   -F_W(-v,mu_X,mu_Y,sigma_X,sigma_Y,rho))-q,
   lower = 0, upper = 1000, maxiter = 1000, extendInt = "yes")
   return(Q$root)}
```

Finally, starting from expression (18), quantiles from $F_{W_a^b}(\cdot)$ can be computed by using the following R-function which has the lower truncation point *a* and the upper truncation point *b* as last arguments.

```
Quantile_W_t<-function(q,mu_X,mu_Y,sigma_X,sigma_Y,rho,a,b){
aa<-F_W(b,mu_X,mu_Y,sigma_X,sigma_Y,rho)+
     -F_W(a,mu_X,mu_Y,sigma_X,sigma_Y,rho)
ks<-q*aa+F_Wl(a,mu_X,mu_Y,sigma_X,sigma_Y,rho)
Q<-Quantile_Wl(ks,mu_X,mu_Y,sigma_X,sigma_Y,rho)
return(Q)}
```

By using these R-functions, all the CIs introduced in Section 5 can be computed.

REFERENCES

Arcagni A. (2014). Zenga Distribution: parameters estimation with constraints on synthetic inequality indices. *Statistica & Applicazioni*, **XII**(1), 17-38.

Arcagni A., Porro F. (2013). On the parameters of Zenga distribution. *Statistical Methods and Applications*, **22**(3), 285-303.

Arcagni A., Porro F. (2016). A comparison of income distributions models through inequality curves. *Statistica & Applicazioni*, **XIV**(2), 123-144.

Arcagni A., Zenga M.M. (2013). Applications of Zenga's distribution to a panel survey on household incomes of European member states. *Statistica & Applicazioni*, **XI**(1), 79-102.

Aroian L.A. (1986). The distribution of the quotient of two correlated random variables. *Proceedings of the Am. Stat. Ass. Business and Economic Section.*

Casella G., Berger R.L. (2002). *Statistical Inference*, 2nd ed. Pacific Grove, CA, Duxbury.

Cochran W.G. (1977). *Sampling Techniques*. J. Wiley, New York.

Chotikapanich D. (Editor) (2008). *Modeling Income Distributions and Lorenz Curves*. Economic Studies in Inequality, Social Exclusion and Well-Being, Springer.

De Capitani L. (2012). Interval Estimation for the Sharpe Ratio when returns are not i.i.d. with special emphasis on the $GARCH(1,1)$ process with symmetric innovations. *Statistical Methods & Applications*, **21**(4), 517-537.

De Capitani L. (2014). Interval Estimation for the Sortino Ratio and the Omega Ratio. *Communications in Statistics - Simulation and Computation,* **43**(6), 1385-1429.

De Capitani L., Pasquazzi L. (2015). Inference for performance measures for financial assets. *Metron,* **73**(1), 73-98.

De Capitani L., Zenga M. (2011). Point and interval estimation for some financial performance measures. *Statistica & Applicazioni*, **IX**(2), 129-158.

De Capitani L., Zini A. (2013a). The Confluent Hypergeometric-Mixture of Polisicchio Distributions: a Generalized Zenga Distribution. *Statistica & Applicazioni*, **XI**(1), 23-48.

De Capitani L., Zini A. (2013b). A comparison among two generalized Beta-mixtures of Polisicchio distributions and the Zenga Model. *Statistica & Applicazioni*, **XI**(2), 193-230.

Fieller E.C. (1932). The distribution of the index in a Normal Bivariate population. *Biometrika,* **24**(3/4), 428-440.

Fieller E.C. (1940). The biological standardisation of insulin. *Journal of the Royal Statistical Society* (Supplement), **7**(1), 1-64.

Fieller E.C. (1954). Some Problems in Interval Estimation. *Journal of the Royal Statistical Society*, Series B (Methodological), **16**(2), 175-185.

Frosini B.V. (1970). La stima di un quoziente nei grandi campioni. *Giornale degli Economisti e Annali di Economia*, Nuova Serie, **29**(5/6), 381-400.

Galeone C. (2007). *On the ratio of two Normal r.v., jointly distributed as a Bivariate Normal*. PhD Thesis, Università di Milano-Bicocca.

Galeone C., Pollastri A. (2009). Sample size for estimating the ratio of two means. In *First italian Conference on Survey Methodology*, Tipografia Senese, 201-204.

Galeone C., Pollastri A. (2012). Confidence intervals for the ratio of two means using the distribution of the quotient of two normals. *Statistics in Transition,* **13**(3), 451-472.

Galeone C., Pollastri A. (2013). Cost-effectiveness Ratio for comparing social and health policies. *Silesian Statistical Review,* **11**(17), 123-131.

Geary R.C. (1930). The frequency distribution of the quotient of two normal variates. *Journal of the Royal Statistical Society,* **93**(3), 442-446.

Genz A., Bretz F., Miwa T., Mi X., Leisch F., Scheipl F., Hothorn T. (2017). mvtnorm: Multivariate Normal and t Distributions. R package version 1.0-6. URL http://CRAN.R-project.org/package=mvtnorm

Hall A.R. (2005). *Generalized Method of Moments*, Oxford University Press.

Hinkley D.V. (1969). On the ratio of two correlated normal random variables. *Biometrika,* **56**(3), 635-639.

Hoeffding W. (1948). A class of statistics with asymptotically normal distribution. *The Annals of Mathematical Statistics*, **19**, 293-325.

Jobson J., Korkie B. (1981). Performance Hypothesis Testing with the Sharpe and Treynor Measure. *The Journal of Finance*, **36**(4), 889-908.

Kotz S., Balakrishnan N., Johnson N.L. (2004). *Continuous Multivariate Distributions*. Wiley, New York.

Lo A.W. (2002). The Statistics of Sharpe Ratio. *Financial Analyst Journal*, **58**, 36-52.

Marsaglia G. (1965). Ratios of normal variables and ratios of sums of uniform variables. *Journal of the American Statistical Association,* **60**(309), 193-204.

Marsaglia G. (2006). Ratios of normal variables. *Journal of Statistical Software,* **16**(4), 1-10.

Miller R., Gehr A. (1978). Sample Bias and Sharpe's Performance Measure: A Note. *Journal of Financial and Quantitative Analysis*, **13**(5), 943-946.

Newey W.K., West K.D. (1987). A Simple, Positive Semi-Definite, Heteroskedasticity and Autocorrelation Consistent Covariance Matrix. *Econometrica*, **55**(3), 703-708.

Newey W.K., West K.D. (1994). Automatic lag selection in covariance matrix estimation, *Review of Economic Studies*, **61**(4), 631-653.

Nicholson C. (1943). The probability integral for two variables. *Biometrika,* **33**(1), 59-72.

Öksoy D., Aroian L.A. (1986). Computational Techniques and examples of the density and the distribution of the quotient of two correlated normal variables. *Proceeding of the American Statistical Association, Business and Economic Section,* **33**, 660-663.

Öksoy D., Aroian L.A. (1994). The quotient of two correlated normal variables with applications. *Communications in Statistics - Simulation and Computation,* **23**(1), 223-241.

Opdyke J.D. (2007). Comparing Sharpe Ratios: So Where are the p-values? *Journal of Asset Management*, **8**(5), 308-336.

Owen D.B. (1956). Tables for computing Bivariate Normal Probabilities. *Annals of Mathematical Statistics,* **27**, 1075-1090.

Pollastri A., Maffenini W. (2018). Estimation of the percentage of aid received from the Food Bank using a Stratified Sampling. *Social Indicators Research,* **136**(1), 41-49.

Pollastri A., Tulli V. (2015). The Distribution of the Absolute Value of the Ratio of two correlated Normal Random Variables. *Statistica & Applicazioni,* **XIII**(1), 107-119.

Porro F. (2015). Zenga Distribution and Inequality Ordering. *Communications in Statistics - Theory and Methods,* **44**(18), 3967-3977.

R Core Team (2017). R: A language and environment for statistical computing. R Foundation for Statistical Computing, Vienna, Austria. URL https://www.R-project.org/.

Serfling R.J. (1980). *Approximation Theorems of Mathematical Statistics*, Wiley.

Sharpe W.F. (1964). Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk. *The Journal of Finance*, **19**, 425-442.

Sharpe W.F. (1966). Mutual fund Performance. *Journal of Business*, **January**, 119-138.

Sharpe W.F. (1994). The Sharpe Ratio. *Journal of Portfolio Management*, **Fall**, 49-58.

Sheppard W.F. (1900). On the calculation of the double-integral expressing normal correlation. *Transaction of the Cambridge Philosophical Society,* **19**, 23-36.

Zenga M.M. (1998). *Introduzione alla Statistica Descrittiva*. Vita e Pensiero, Milano.

Zenga M.M. (2010). Mixture of Polisicchio's truncated Pareto distributions with beta weights. *Statistica & Applicazioni*, **VIII**(1), 3-25.

Zenga M.M., Pasquazzi L., Polisicchio M., Zenga Ma. (2011). More on M.M. Zenga's new three-parameter distribution for non-negative variables, *Statistica & Applicazioni*, **IX**(1), 5-33.

Zenga M.M., Pasquazzi L., Zenga Ma. (2012). First applications of a new three-parameter distribution for non-negative variables. *Statistica & Applicazioni*, **X**(2), 131-147.