Statistica & Applicazioni - Vol. XVII, n. 1, 2019, p. 35-62

INCREASING THE EFFICIENCY OF RANKED SET SAMPLING VIA VISUAL GROUPING WITH RESPECT TO A THRESHOLD

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SUMMARY

In this paper, a new sampling technique is proposed that carries more information than contained in ranked set sample (RSS). The proposed sampling technique is defined by making the use for the idea of visual grouping of population units with respect to a fixed threshold and RSS. We refer to it as RSS-Grouping technique. Under the best informative RSS-Grouping technique, the maximum likelihood estimator (MLE) of the mean of an exponential distribution is derived. This MLE is then compared to various candidate estimators through extensive simulation experiments. Numerical results show that the MLE under the best informative RSS-Grouping scheme is preformed better than these estimators. The effects of imperfect sampling on the behavior of the MLE under the proposed scheme is also studied. We conduct a simulation study to assess the finite sample behavior of the MLE under imperfect sampling and imperfect classification of visual grouping. Similarly, the simulation study shows that the MLE is behaved asymptotically unbiased. Additionally, the MLE tends to be at least as efficient as the MLE under RSS regardless of raking errors and the estimation of the threshold has slightly effects on the sampling distribution of the MLE.

Keywords: *Exponential Distribution, Maximum Likelihood, Ranked Set Sampling, Simple Random Sample, Threshold.*
DOI: 10.26350/999999_000021
ISSN: 18246672 (print)

1. INTRODUCTION

In many practical situations such as environmental, ecological, and agricultural studies, one may be interested in making inferences about some important unknown quantities from these studies. Often, measurements of the variable of interest for observed units from these studies are expensive or time-consuming, see for example,

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Chen, Bai and Sinha (2004), Cobby, Ridout, Bassett and Large (1985), Dell and Clutter (1972). In many situations, the available sample does not provide sufficient information, and at the same time, the researcher is unable to obtain further units due to cost and time issues. To overcome this difficulty, McIntyre (1952) introduced ranked set sample that is capable of handling problems arising in such similar circumstances effectively. RSS is a sampling technique which utilizes the experience of the experimenter to rank a small number of the population units with respect to the variable of interest via the visual ranking so as to obtain more information than the classical simple random sampling (SRS) with no additional cost. For example, McIntyre (1952) showed in his study of estimating the mean yield pasture that visual ranking of units, without quantifying them, can improve the coverage of a sample. For more recent information about theory, applications and ramifications of ranked set sampling methods, we refer the reader to the comprehensive survey book by Chen *et al.*, (2004) and the references therein. According to McIntyre (1952), RSS is defined as follows.

Step 1. Select a random sample of size m^2 from the target population

Step 2. Randomly allocate the m^2 selected units into m sets each of size m

Step 3. Use a visual ranking or any cost free method to rank the units within each set with respect to the variable of interest.

Step 4. Choose for actual measurement the j^{th} order statistics in the i^{th} set, where $i,j = 1, 2, , m$. That is, the RSS consists of selecting the smallest ranked unit in the first set, followed by the second smallest ranked unit in the second set, continuing in this way till the largest ranked unit is selected in the last set.

Step 5. Repeat Steps 1 through 4 r times (cycles) to get a sample of size $n = mr$

In symbols, let $X_{i,j}$ the j^{th} be value of the variable of interest in the i^{th} set for $i,j = 1,\ 2,\ ,m$. In matrix notation, the data can be organized as follows:

$$\mathbf{X} = \begin{pmatrix} X_{1,1} & X_{1,2} & \cdots & X_{1,m-1} & X_{1,m} \\ X_{2,1} & X_{2,2} & \cdots & X_{2,m-1} & X_{2,m} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ X_{m-1,1} & X_{m-1,2} & \cdots & \ddots & X_{m-1,m} \\ X_{m,1} & X_{m,2} & \cdots & X_{m,m-1} & X_{m,m} \end{pmatrix}.$$

After applying the ranking procedure to the raw data inserted in the matrix $\mathbf{X}$, we obtain the ranked data given in the following matrix whose entries are represented by $X_{i,(j)}$, the j^{th} order statistics in the i^{th} set, for $i,j = 1,\ 2,\ ,m,$

$$\mathbf{Y} = \begin{pmatrix} \mathbf{X_{1,(1)}} & X_{1,(2)} & \cdots & X_{1,(m-1)} & X_{m,(m)} \\ X_{2,(1)} & \mathbf{X_{2,(2)}} & \cdots & X_{2,(m-1)} & X_{2,(m)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ X_{m-1,(1)} & X_{m-1,(2)} & \cdots & \ddots & X_{m-1,(m)} \\ X_{m,(1)} & X_{m,(2)} & \cdots & X_{m,(m-1)} & \mathbf{X_{m,(m)}} \end{pmatrix}.$$

According to Step 4, the units on the diagonal of $\mathbf{X_{m,(m)}}$, $\mathbf{X_{1,(1)}}$, $\mathbf{X_{2,(2)}}$, $\ldots$, $\mathbf{X_{m,(m)}}$, constitute the cycle RSS for one cycle. For simplicity, the notation Y_1, $Y_2, \ldots, Y_m$ is used instead of $\mathbf{X_{1,(1)}}$, $\mathbf{X_{2,(2)}}$, $\ldots$, $\mathbf{X_{m,(m)}}$. When Step 5 is applied, then $\left\{Y_{ij},\ \ i = 1, \ldots, m; j = 1, \ldots, r\right\}$ would have formed a ranked set sample of size $n = mr$, where again Y_{ij} represents the actual measurement for the unit in i^{th} set and j^{th} cycle.

The McIntyre's RSS has been modified by several authors in order to improve the accuracy of inference and in particular when the underlying distribution is parametric. For example, Samawi, Ahmed and Abu-Dayyeh (1996) introduced the extreme RSS to estimate the mean of a symmetric population. Öztürk and Wolfe (2000) investigated the effect of different ranked set sampling protocols such as the sequential, mid-range and fixed sampling design on the sign test statistic. In a similar context, Öztürk and Wolfe (2001) constructed a new ranked set sampling protocol that maximizes the Pitman asymptotic efficiency of the singed test. Additionally, Öztürk (2002) developed a rank-based estimator and testing procedures for linear models based on ranked-set samples. In particular, he showed that the estimator of the regression parameter is asymptotically normal and has higher Pitman asymptotic efficiency than a simple random-sample rank regression estimator.

While the sample obtained under the ranked set sample protocol assumes that the experimenters judgment is perfect, in practice, however, the effect of judgment in the ranking of the sample need not be completely accurate which yield to what so called ranking errors. So, attempts have been made in the literature in order to study the effects of ranking errors on estimation problems under various models, see for example, Bohn and Wolfe (1994), Dell and Clutter (1972), Hatefi and Jozani (2013), MacEachern, Öztürk, Wolfe and Stark (2002), Nahhas, Wolfe and Chen (2004), Wang Y-G, Chen and Liu (2004), Presnell and Bohn (1999). The general results, in these cited papers, indicate that regardless of the quality of ranking (whether is perfectly or imperfectly accurate), statistical procedures based on RSS behave at least as efficient as procedures based on SRS. In particular, Dell and Clutter (1972) was the first who investigated theoretically the problem of raking errors and who have demonstrated that the RSS estimator of the a population mean is unbiased estimator and is at least as efficient as the SRS estimator with same number of quantification regardless of the raking errors. Frey (2014) provided an example where Fisher information number contained in an imperfect ranked set sample can be higher than the Fisher information contained in a perfect ranked set sample. Al-Rawwash, Alodat, Aludaat, Alodat and Muhaidat (2010) studied the effect of ranking errors on the length of the prediction intervals from normal distribution using a ranking errors model proposed by Dell and Clutter (1972). In this paper we propose a new sampling technique by using the idea of visual grouping of population with respect to a threshold coefficient value along with RSS. Our motivation of introducing this technique is that the proposed scheme carries further information than RSS with negligible cost and to improve statistical inference.

The rest of the paper is organized as follows. In Section 2, the proposed sampling technique is presented. The MLE of the parameter of the exponential distribution under the proposed procedure is given in Section 3. In Section 4, we discuss the best informative RSS scheme for the exponential distribution and we study the proportion of information added by visual grouping. In Section 5, we derive the maximum likelihood estimators of the parameter of the exponential distribution for samples obtained via the best informative RSS and the the best informative RSS-Grouping. In Section 6, we conduct a simulation study to compare between these estimators in terms of their biases, mean square errors and efficiencies. In Section 7, we study the behavior of the maximum likelihood estimator (MLE) via a simulation study under imperfect sampling, imperfect classification of visual grouping, and when the threshold is unknown. Conclusions and some suggested open problems are given in Section 8.

2. THE PROPOSED SAMPLING TECHNIQUE

It is important to point out that although RSS technique requires identification of m^2 units, only m of these m^2 units are considered for actual quantification. So, RSS should be compared to an SRS of the same size. In SRS, the experimenter must increase the sample size in order to get a sample of more coverage for the population range, whereas in RSS the experimenter can utilize the idea of visual ranking to give a sample with better coverage for the population range compared to the SRS of the same size. This leads also to a saving in sampling cost. In this paper, we propose to use the idea of visual grouping of units together with RSS sampling technique in order to add information with negligible cost to the sample. Our procedure is described as follows:

Step 1. Select a RSS sample of size $n = mr$ as described in Section 1 by Steps 1 through 5. Let$Y_{ij}, i = 1, m,\ j = 1, \dots,\ r$ denote the selected RSS sample.

Step 2. Select another random sample of size rm^2 units from the population of interest.

Step 3. Randomly allocate the selected random sample in Step 2, into r blocks so that there is m^2 in each block.

Step 4. Use the visual method or any other cheap method to classify the units in the i^{th} set (without quantifying them) with respect to the characteristic of interest, as above (below) a single fixed known threshold T. Let Z_{ij} be the number of units in the i^{th} row of the j^{th} block that are detected, via visual ranking, to be above (below) the threshold T. As result, the above procedure will produced a sample, $\mathcal{D} = \{Y_{ij},\ Z_{ij}; i = 1, 2, \dots m,\ j = 1, \dots r\}$ which consists of two parts, namely $\mathbf{Y} = \left\{Y_{ij};\ i = 1, \dots, m,\ j = 1, \dots, r\right\}$, the ranked set sample and the additional (free or negligible cost) sample $\mathbf{Z} = \left\{Z_{ij};\ i = 1, \dots, m, j = 1, \dots, r\right\}$.

The idea beyond the above proposed technique is that: it provides additional information to the RSS without quantifying more units, i.e., it is equivalent to RSS of size $n = mr$ in terms of the cost. More specifically, there is negligible cost added to the ranked set sample when collecting relatively small number of the observations $Z_{ij},\ i = 1, 2, \ldots m,\ j = 1, 2, \ldots, r$. Furthermore we may think of the procedure of collecting the $Z'_{ij}s$ as grouping of selected data via a visual mechanism. While RSS can be thought of as random stratification, the above technique divides the population into two non-random strata (above or below T). We refer to the sampling scheme described by Steps 1 through 4 as RSS-Grouping sampling. It may be argued that how should the thresholding coefficient value be determined since this value usually unknown in practice. Here we provide two different ways that the threshold T can be chosen.

i) It can be chosen according to an experimenter's opinion or from previous studies. For example, when the researcher is interested in estimating the average yield of olives trees in a farm, he can use for T any value between the average yield per tree in the ''off'' year and the average yield per tree of the ''on'' year. It is observed by Barone, La Mantia, Marchese, and Marra (2014) that the average yield is 21.6 kg per tree in 2008, the ''on'' year, versus 8.25 kg per tree in 2009, the ''off'' year. So any value between 8.25 and 21.6 is a valid candidate for the value of T.
ii) The κ^{th} percentile of the population can be used as a proposal for T, where $\kappa \in (0, 1)$. In this case, T will be a parameter and hence can be estimated using the non-grouping part of the RSS-Grouping sample.

3. LIKELIHOOD INFERENCE FROM RSS-GROUPING

Consider a population described by a distribution function $F(x; \theta)$ with probability density function $f(x; \theta), \theta \in \Theta$. Suppose that $\{f(x; \theta), \theta \in \Theta\}$ satisfies the regularity conditions given in Frey (2014). If $\mathbf{Y} = \{Y_{ij}; i = 1, \ldots m, j = 1, \ldots, r\}$, $\mathbf{Z} = \{Z_{ij};\ i = 1, \ldots m,\ j = 1, \ldots, r\}$, and $\mathcal{D} = \{Y_{ij},\ Z_{ij};\ i = 1, \ldots m,\ j = 1, \ldots, r\}$ are the data obtained by applying RSS-Grouping technique, then the likelihood function for $\mathcal{D}$ is:

$$L(\theta; D) = \prod_{i=1}^{m} \prod_{j=1}^{r} f_{Y_{ij}}(y_{ij}; \theta) \prod_{i=1}^{m} \prod_{j=1}^{r} P_{Z_{ij}}\left(Z_{ij} = z_{ij}; \theta\right), \tag{1}$$

where $f_{Y_{ij}}(y_{ij}; \theta)$ is given by

$$f_{Y_{ij}}(y_{ij}; \theta) = \frac{m!}{(i-1)!(m-i)!} F(y_{ij}; \theta)^{i-1} \left(1 - F(y_{ij}; \theta)\right)^{m-i} f(y_{ij}; \theta), \tag{2}$$

and $P_{Z_{ij}}\left(Z_{ij} = z_{ij}; \theta\right)$ is given by

$$P_{Z_{ij}}\left(Z_{ij} = z_{ij}; \theta\right) = \binom{m}{z_{ij}} F(T; \theta)^{m - z_{ij}} (1 - F(T; \theta))^{z_{ij}}. \tag{3}$$

Clearly, (3) shows that $\mathcal{D}$ carries more information about θ than $\mathcal{Y}$, the RSS. In this paper we apply RSS-Grouping to the exponential distribution, i.e., when the parent distribution of X has the pdf

$$f(x;\theta)=\begin{cases}\frac{1}{\theta}e^{-\frac{x}{\theta}}, & x>0,\\ 0, & x<0.\end{cases}$$

On using (1), the log-likelihood function reduces to

$$\log L(\theta;\mathcal{D})=\log C-mr\log\theta+\sum_{j=1}^{r}\sum_{i=1}^{m}(i-1)\log\left(1-e^{-y_{ij}/\theta}\right)-\frac{1}{\theta}\sum_{j=1}^{r}\sum_{i=1}^{m}iy_{ij}$$

$$+\sum_{j=1}^{r}\sum_{i=1}^{m}(m-z_{ij})\log\left(1-e^{-T/\theta}\right)-\frac{T}{\theta}\sum_{j=1}^{r}\sum_{i=1}^{m}z_{ij}, \tag{4}$$

where $C=\prod_{i=1}^{m}\prod_{j=1}^{r}\dfrac{m!}{(i-1)!(m-i)!}\binom{m}{z_{ij}}$. Now differentiate the log-likelihood equation given by (4) with respect to θ to get

$$\frac{\partial\log L(\theta;\mathcal{D})}{\partial\theta}=-\frac{mr}{\theta}-\frac{1}{\theta^2}\sum_{j=1}^{r}\sum_{i=1}^{m}\frac{(i-1)Y_{ij}e^{-\frac{Y_{ij}}{\theta}}}{1-e^{-\frac{Y_{ij}}{\theta}}}+\frac{1}{\theta^2}\sum_{j=1}^{r}\sum_{i=1}^{m}iY_{ij}$$

$$-\frac{Te^{-T/\theta}}{\theta^2(1-e^{-T/\theta})}\sum_{j=1}^{r}\sum_{i=1}^{m}(m-z_{ij})+\frac{T}{\theta^2}\sum_{j=1}^{r}\sum_{i=1}^{m}z_{ij}. \tag{5}$$

Setting the right hand side of (5) to zero and multiplying by $-\theta^2$ to get

$$mr\theta+\sum_{j=1}^{r}\sum_{i=1}^{m}\frac{(i-1)Y_{ij}e^{-\frac{Y_{ij}}{\theta}}}{1-e^{-\frac{Y_{ij}}{\theta}}}-\sum_{j=1}^{r}\sum_{i=1}^{m}iY_{ij}$$

$$+\frac{Te^{-T/\theta}}{(1-e^{-T/\theta})}\sum_{j=1}^{r}\sum_{i=1}^{m}(m-z_{ij})-T\sum_{j=1}^{r}\sum_{i=1}^{m}z_{ij}=0. \tag{6}$$

The solution of (6) in terms of θ yields the maximum likelihood estimator of θ. Since (6) does not have a closed form solution, we obtain a numerical approximation ofthe solution.

4. BEST INFORMATIVE SAMPLING SCHEMES

Since the McIntyre RSS scheme can be improved under parametric families, then we compare the RSS-Grouping sampling to best informative sampling scheme, i.e., the sampling scheme which contains the maximum amount of Fisher information about the parameter θ. If $X\sim f(x;\theta)=\frac{1}{\theta}e^{-x/\theta}$, $x>0;\ \theta>0$, then the Fisher information number about θ contained in $Y_{j:m}$ which is denoted by $\mathcal{I}_{Y_{j:m}}(\theta)$ is given by Arnold, Balakrishnan and Nagaraja (1992) as follows:

$$\theta^2 \mathcal{I}_{Y_{j:m}}(\theta) = C_{j,m} \int_0^\infty \left(\frac{-(j-1)ve^{-v}}{1-e^{-v}} + (m-j+1)v - 1 \right)^2 h(v)dv, \tag{7}$$

where $h(v) = (1-e^{-v})^{j-1} e^{-(m-j+1)v}$ and $C_{j,m} = \frac{m!}{(j-1)!(m-j)!}$.

To find the value of j that maximizes $\mathcal{I}_{Y_{j:m}}(\theta)$ as a function of j, it is equivalent to maximize $\theta^2 \mathcal{I}_{Y_{j:m}}(\theta)$ for j. Let $j_0 = \arg\max_{j=0,\ldots,m} \theta^2 \mathcal{I}_{Y_{j:m}}(\theta)$. It is difficult to find j_0 theoretically, and therefore we find its value for different values of m. Table 1 shows the values of $\theta^2 \mathcal{I}_{Y_{j:m}}(\theta)$ for different values of m and j. Here, we consider cases when $j, m = 1, \ldots, 10$.

Table 1 shows that the amount of information about θ collected by the $Y_{j:m}$ attains its maximum, as a function of j, at

$$j_0 = \begin{cases} m & \textit{if } 1 \le m \le 5 \\ m-1 & \textit{if } 6 \le m \le 10 \end{cases}. \tag{8}$$

TABLE 1. - *The values of $\theta^2 \mathcal{I}_{Y_{j:m}}(\theta)$ for $Exp(\theta)$ for different j and m*

m	j	$\theta^2 \mathcal{I}_{Y_{j:m}}(\theta)$	m	j	$\theta^2 \mathcal{I}_{Y_{j:m}}(\theta)$	m	j	$\theta^2 \mathcal{I}_{Y_{j:m}}(\theta)$
3	1	1.0000	7	1	1.000	9	1	1.000
	2	1.9247		2	1.9883		2	1.9931
	3	2.5000		3	2.9444		3	2.9688
4	1	1.000		4	3.8311		4	3.9101
	2	1.9605		5	4.5739		5	4.7893
	3	2.7778		6	5.0036		6	5.5579
	4	3.1111		7	4.6314		7	6.1229
5	1	1.000	8	1	1.000		8	6.2818
	2	1.9758		2	1.9911		9	5.4723
	3	2.875		3	2.9592	10	1	1.000
	4	3.5694		4	3.8798		2	1.9945
	5	3.6620		5	4.7099		3	2.9753
6	1	1.000		6	5.3654		4	3.9301
	2	1.9837		7	5.6596		5	4.8399
	3	2.9200		8	5.0653		6	5.6734
	4	3.7450	9	1	1.000		7	6.3770
	5	4.3089		2	1.9931		8	6.8497
	6	4.1658		3	2.9688		9	6.8741
							10	5.8561

For $1 \le m \le 5$, the best informative RSS sampling scheme is the one which chooses the maximum of each set for actual quantification.While for $6 \le m \le 10$, we choose $Y_{m-1:m}$ for actual quantification. Notice that the best informative RSS scheme is defined to be $(j_0, \ldots, j_0)_m$, where j_0 is given by (8). In Table 2 we display the best informative RSS scheme and the corresponding amount of Fisher information of θ. Similarly, the amount of Fisher information about θ that added due to using visual grouping is calculated as follows. Since $Z_{ij} \sim Bin(m,\ \beta)$, where $\beta = 1 - F(T;\theta)$, the Fisher information number about β contained in Z_{ij} see, Frey (2014), is:

$$\mathcal{I}_{Z_{ij}}(\beta) = \frac{m}{\beta(1-\beta)}.$$

TABLE 2. - *The values of Fisher information for Exp(θ) in the best RSS plan for different m*

m	Best informative RSS plan	Fisher Information in plan
3	(3,3,3)	7.5000
4	(4,4,4,4)	12.4444
5	(5,5,5,5,5)	18.3102
6	(5,5,5,5,5,5)	25.8533
7	(6,6,6,6,6,6,6)	35.0253
8	(7,7,7,7,7,7,7,7)	45.2770
9	(8,8,8,8,8,8,8,8,8)	56.5361
10	(9,9,9,9,9,9,9,9,9,9)	68.7407

Since $\theta = g(\beta) = -T/\log(1-\beta)$, then the Fisher information about θ contained in Z_{ij} is

$$\mathcal{I}_{Z_{ij}}(\theta) = \frac{\mathcal{I}_{Z_{ij}}(\beta)}{(g'(\beta))^2}.$$

Hence the Fisher information number about θ contained in $\mathcal{Z}$ is

$$\mathcal{I}_{\mathcal{Z}}(\theta) = \frac{r\mathcal{I}_{Z_{ij}}(\beta)}{(g'(\beta))^2} = \frac{mrT^2}{\theta^4} \times \frac{1-e^{-T/\theta}}{e^{-T/\theta}}.$$

So the proportion of information added by $\mathcal{Z}$ to the best informative scheme is

$$P(m,r,j_0,T,\theta) = \frac{\mathcal{I}_{\mathcal{Z}}(\theta)}{r\mathcal{I}_{Y_{j_0:m}}(\theta) + \mathcal{I}_{\mathcal{Z}}(\theta).}$$

Notice that $P(m,r,j_0,T,\theta)$ is free of r. Table 3 presents values of $P(m,r,j_0,T,\theta)$ for different values of, m, j_0, θ and T. Table 3 shows that the proportion $P(m,r,j_0,T,\theta)$ is an increasing function of T for fixed values of m, j_0, θ. Moreover, regardless of the value of m, the proportion $P(m,r,j_0,T,\theta)$ is increasing function of m.

TABLE 3. - *Proportion of added information by visual ranking for θ, T and different m*

θ	$T=1$			$T=4$			$T=7$		
3	m	j_0	$P(m,r,j_0,T,\theta)$	m	j_0	$P(m,r,j_0,T,\theta)$	m	j_0	$P(m,r,j_0,T,\theta)$
	3	3	0.0501	3	3	0.8563	3	3	0.9838
	4	4	0.0536	4	4	0.8646	4	4	0.9850
	5	5	0.0566	5	5	0.8714	5	5	0.9858
	6	5	0.0577	6	5	0.8737	6	5	0.9860
	7	6	0.0579	7	6	0.8742	7	6	0.9861
	8	7	0.0585	8	7	0.8753	8	7	0.9862
	9	8	0.0593	9	8	0.8768	9	8	0.9864
	10	9	0.0601	10	9	0.8784	10	9	0.9866
	$T=5$			$T=9$			$T=15$		
10	m	j_0	$P(m,r,j_0,T,\theta)$	m	j_0	$P(m,r,j_0,T,\theta)$	m	j_0	$P(m,r,j_0,T,\theta)$
	3	3	0.1629	3	3	0.5866	3	3	0.9039
	4	4	0.1725	4	4	0.6032	4	4	0.9097
	5	5	0.1813	5	5	0.6175	5	5	0.9145
	6	5	0.1842	6	5	0.6221	6	5	0.9160
	7	6	0.1849	7	6	0.6232	7	6	0.9164
	8	7	0.1865	8	7	0.6256	8	7	0.9172
	9	8	0.1886	9	8	0.6288	9	8	0.9182
	10	9	0.1909	10	9	0.6323	10	9	0.9193
	$T=10$			$T=16$			$T=20$		
15	m	j_0	$P(m,r,j_0,T,\theta)$	m	j_0	$P(m,r,j_0,T,\theta)$	m	j_0	$P(m,r,j_0,T,\theta)3$
	3	3	0.3358	3	3	0.7224	3	3	0.8563
	4	4	0.3514	4	4	0.7360	4	4	0.8646
	5	5	0.3651	5	5	0.7475	5	5	0.8715
	6	5	0.3697	6	5	0.7512	6	5	0.8737
	7	6	0.3708	7	6	0.7521	7	6	0.8742
	8	7	0.3732	8	7	0.7540	8	7	0.8753
	9	8	0.3764	9	8	0.7565	9	8	0.8768
	10	9	0.3700	10	9	0.7593	10	9	0.8784

5. ESTIMATION BASED ON BEST INFORMATIVE RSS SCHEME

In this section, we use data collected via the best informative RSS scheme and the visual grouping to derive estimators to the parameter θ. Two estimation methods will be considered, the MLE and the unbiased estimator.

5.1 *Maximum Likelihood Estimator*

Let $Y^{(1)}_{j_0:m}, Y^{(2)}_{j_0:m}, \ldots, Y^{(r)}_{j_0:m}$ be i.i.d copies from $Y_{j_0:m.}$ The pdf of $Y^{(i)}_{j_0:m},\ i = 1,\ 2,\ \ldots,\ r$, is

$$g_{Y_{j_0:m}}(y;\theta) = C_{j_0,m}\left(1 - e^{-\frac{y}{\theta}}\right)^{j_0-1} e^{-\frac{(m+1-j_0)y}{\theta}}\ \frac{1}{\theta}.$$

and the log likelihood function is given by

$$\log\left(g_{Y_{j_0:m}}(y;\theta)\right) = r\log(C_{j_0,m}) - r\log\theta + (j_0 - 1)\sum_{i=1}^{r}\log\left(1 - e^{-\frac{Y^{(i)}_{j_0:m}}{\theta}}\right) - K_{j_0:m}\ \frac{\sum_{i=1}^{m} Y^{(i)}_{j_0:m}}{\theta}, \tag{9}$$

where $K_{j_0:m} = m - j_0 + 1$. Taking the first derivative of (9) with respect to θ gives

$$\frac{\partial \log g_{Y_{j_0:m}}(y;\theta)}{\partial\theta} = -\frac{r}{\theta} + K_{j_0:m}\frac{\sum_{i=1}^{r} Y^{(i)}_{j_0:m}}{\theta^2} - \frac{(j_0-1)}{\theta^2}\sum_{i=1}^{r}\frac{Y^{(i)}_{j_0:m}\ e^{-\frac{Y^{(i)}_{j_0:m}}{\theta}}}{1 - e^{-\frac{Y^{(i)}_{j_0:m}}{\theta}}}. \tag{10}$$

Setting (10) equal zero and solving for θ yields the equation

$$(m - j_0 + 1)\sum_{i=1}^{r} Y^{(i)}_{j_0:m} - (j_0 - 1)\sum_{i=1}^{r}\frac{Y^{(i)}_{j_0:m}e^{-\frac{Y^{(i)}_{j_0:m}}{\theta}}}{1 - e^{-\frac{Y^{(i)}_{j_0:m}}{\theta}}} = r\theta. \tag{11}$$

The MLE of θ, under the best informative RSS, is the solution of (11). Similarly, the MLE of θ, under the RSS-Grouping scheme, is obtained by solving the following equation:

$$(m - j_0 + 1)\sum_{j=1}^{r} Y^{(j)}_{j_0:m} - (j_0 - 1)\sum_{j=1}^{r}\frac{Y^{(j)}_{j_0:m}e^{-\frac{Y^{(j)}_{j_0:m}}{\theta}}}{1 - e^{-\frac{Y^{(j)}_{j_0:m}}{\theta}}} - \frac{Te^{-\frac{T}{\theta}}}{1 - e^{-\frac{T}{\theta}}}\sum_{j=1}^{r}\left(m - z_j\right) + rT\bar{z} = r\theta, \tag{12}$$

where $z_1,\ \ z_2, \ldots, z_r$ are independent Binomial(m, β) random variables and $\bar{z}$ is the sample mean. It is clear that no closed form solutions to (11) and (12) are available and therefore numerical solutions will be obtained via iteration methods.

5.2 *Unbiased Estimators based on best informative RSS scheme*

Let $(j_0, \ldots, j_0)_m$ be the best informative RSS scheme for Exp(θ). Then the pdf of $Y_{j_0:m}$ is

$$g_{j_0}(y;\theta) = C_{j_0,m}\left(1 - e^{-\frac{y}{\theta}}\right)^{j_0-1} e^{-(m-j_0+1)\frac{y}{\theta}}\ \frac{1}{\theta}\ ; \qquad y > 0,$$

where $C_{j_0,m} = \frac{m!}{(j_0-1)!(m-j_0)!}$. The mean of $Y_{j_0:m}$ is

$$E\left(Y_{j_0:m}\right) = C_{j_0,m}\int_0^\infty \frac{y}{\theta}\left(1 - e^{-\frac{y}{\theta}}\right)^{j_0-1} e^{-(m-j_0+1)\frac{y}{\theta}}\, dy = \theta\, A_{mj_0},$$

where

$$A_{mj_0} = C_{j_0,m}\int_0^\infty v(1 - e^{-v})^{j_0-1} e^{-(m-j_0+1)v}\, dv.$$

An unbiased estimator of θ based on $Y^{(1)}_{j_0:m}, Y^{(2)}_{j_0:m}, \ldots,\ Y^{(m)}_{j_0:m}$, m independent copies of $Y_{j_0:m}$, is

$$\widehat{\theta}_4 = \frac{1}{mA_{mj_0}}\sum_{j=1}^{m} Y^{(j)}_{j_0:m} \qquad \text{with} \qquad Var\left(\widehat{\theta}_4\right) = \frac{1}{mA^2_{mj_0}}\ Var\left(Y^{(1)}_{j_0:m}\right),$$

where

$$Var\left(Y^{(1)}_{j_0:m}\right) = E\left({Y^{(1)}_{j_0:m}}^2\right) - \left(EY^{(1)}_{j_0:m}\right)^2 =$$

$$= C_{j_0,m}\int_0^\infty v^2(1 - e^{-v})^{j_0-1} e^{-(m-j_0+1)v}\, dv -$$

$$-\left(C_{j_0,m}\int_0^\infty v(1 - e^{-v})^{j_0-1} e^{-(m-j_0+1)v}\, dv\right)^2.$$

Under the McIntyre's RSS, the estimator

$$\widehat{\theta}_5 = \frac{1}{m}\sum_{j=1}^{m}\frac{Y_{j:m}}{A_{mj}} \qquad \text{with} \qquad Var\left(\widehat{\theta}_5\right) = \frac{1}{m^2}\sum_{j=1}^{m}\frac{\sigma^2_{j:m}}{A^2_{mj}},$$

is unbiased for θ, where

$$\sigma^2_{j:m} = \int_0^\infty y^2 C_{j,m}\left(1 - e^{-\frac{y}{\theta}}\right)^{j-1} e^{-(m-j+1)\frac{y}{\theta}}\ \frac{1}{\theta}\, dy -$$

$$-\left(\int_0^\infty yC_{j,m}\left(1-e^{-\frac{y}{\theta}}\right)^{j-1} e^{-(m-j+1)\frac{y}{\theta}}\ \frac{1}{\theta}\ dy\right)^2 =$$

$$= C_{j,m}\int_0^\infty v^2(1-e^{-v})^{j-1}e^{-(m-j+1)v}\ dv$$

$$-\left(C_{j,m}\int_0^\infty v(1-e^{-v})^{j-1}e^{-(m-j+1)v}\ dv\right)^2.$$

The efficiency of $\widehat{\theta}_4$ with respect to $\widehat{\theta}_5$

$$eff\left(\widehat{\theta}_5,\ \widehat{\theta}_4\right) = \frac{A^2_{mj_0}\sum_{j=1}^m\left(\sigma^2_{j:m}/A^2_{mj}\right)}{m\ Var\left(Y^{(1)}_{j_0:m}\right)}. \quad (13)$$

6. COMPARING ESTIMATORS

In this section, we conduct a simulation study to compare among the estimators defined in the previous sections. The estimators are compared in terms of their biases, mean squared errors (MSE), and efficiencies. Recall that if V and W are candidate estimators for some unknown parameter say ψ, then the efficiency V with respect to W is defined as

$$eff(V,W) = \frac{MSE(V)}{MSE(W)}.$$

If $eff(V,W) > 1$ then the estimator W is more efficient than the estimator$V$. The following estimators will be used:
$\widehat{\theta}_1$: MLE based on McIntyre RSS scheme
$\widehat{\theta}_2$: MLE based on best informative RSS scheme, i.e., the solution of (11)
$\widehat{\theta}_3$: MLE based on best RSS-Grouping scheme i.e., the solution of (12)
$\widehat{\theta}_4$: Unbiased estimator based on best RSS
$\widehat{\theta}_5$: Unbiased estimator based on McIntyre RSS

Estimators $\widehat{\theta}_4$ and $\widehat{\theta}_5$ are compared in terms of their efficiencies using (13) that are presented in Table 4 for different values of m.

TABLE 4. - *The efficiency of $\widehat{\theta}_4$ with respect to $\widehat{\theta}_5$ for different m*

m	$eff\left(\widehat{\theta}_5,\ \widehat{\theta}_4\right)$	m	$eff\left(\widehat{\theta}_5,\ \widehat{\theta}_4\right)$
1	1.000	6	1.7471
2	1.3500	7	1.6898
3	1.5091	8	1.8163
4	1.4430	9	1.9159
5	1.6223	10	1.8637

Clearly Table 4 illustrates that $eff\left(\widehat{\theta}_5,\ \widehat{\theta}_4\right) \geq 1$ for all $m = 1,\ 2,\ ,10$. So $\widehat{\theta}_4$ is more accurate than $\widehat{\theta}_5$ as an estimator for θ. Next, for an estimator $\widehat{\theta} \in \left\{\widehat{\theta}_1,\ \widehat{\theta}_2, \widehat{\theta}_3\right\}$, let $\widehat{\theta}_i$ denote the estimate of θ obtained from the i^{th} simulated sample. If L samples are generated then the bias and the MSE of the estimator $\widehat{\theta}$ are approximated by: $\text{Bias}\left(\widehat{\theta}\right) \approx \frac{1}{L}\sum_{i=1}^{L}\left(\widehat{\theta}_i\ - \theta\right)$ and $\text{MSE}\left(\widehat{\theta}\right) \approx \frac{1}{L}\sum_{i=1}^{L}\left(\widehat{\theta}_i\ - \theta\right)^2$. The simulation algorithm was carried out $L = 10,000$ times for different values m and j_0. Table 5 demonstrates that estimator $\widehat{\theta}_1$ has smaller bias than $\widehat{\theta}_2$, while estimator $\widehat{\theta}_2$ has smaller MSE than $\widehat{\theta}_1$. Clearly biases of both estimators are decaying to zero. All efficiencies are greater than one and this implies that, under the best informative sampling scheme, the MLE is performed better than its counterpart under the McIntyre's RSS. Therefore, it is reasonable to compare the MLEs $\widehat{\theta}_2$ and $\widehat{\theta}_3$ based on the best informative sampling instead of the McIntyre's RSS.

TABLE 5. - *Bias and MSE for $\widehat{\theta}_1$ and $\widehat{\theta}_2$ and Efficiency $\widehat{\theta}_2$ with respect to $\widehat{\theta}_1$*

				$\widehat{\theta}_1$		$\widehat{\theta}_2$		
m	k	r	j_0	BIAS	MSE	BIAS	MSE	$eff(\widehat{\theta}_1, \widehat{\theta}_2)$
3	5	15	3	0.0064	0.3297	0.0119	0.2388	1.3809
3	10	30	3	0.0028	0.1647	0.0069	0.1190	1.3337
3	15	45	3	0.0008	0.1117	0.0049	0.0807	1.3838
3	20	60	3	0.0024	0.0861	-0.0013	0.0582	1.4795
5	3	15	5	0.0020	0.2277	0.0177	0.1733	1.3138
5	6	30	5	0.0108	0.1163	0.0024	0.089	1.3062
5	9	45	5	0.0028	0.0764	0.0037	0.0551	1.3881
5	12	60	5	0.0070	0.0569	0.0019	0.0428	1.3282

Table 6 presents biases, MSEs, and efficiencies of $\widehat{\theta}_2$ and $\widehat{\theta}_3$ for $\theta = 3,\ T = 1.5$ for different values of $m,\ r$ and j_0.

Estimator $\widehat{\theta}_2$ has smaller bias than the estimator $\widehat{\theta}_3$ but both have biases that are decreasing to 0 and this establishes that $\widehat{\theta}_2$ and $\widehat{\theta}_3$ are behaved asymptotically unbiased. The MSE of $\widehat{\theta}_3$ is smaller than the MSE of $\widehat{\theta}_2$. Furthermore both biases are decaying to zero. Moreover, all values of efficiencies are greater than one.

TABLE 6. - *Bias, MSE and efficiency of the MLE estimator of θ based on two sample schemes for $\theta = 3$ and $T = 1.5$*

			$\widehat{\theta}_2$		$\widehat{\theta}_3$		
r	m	j_0	BIAS	MSE	BIAS	MSE	$eff(\widehat{\theta}_2, \widehat{\theta}_3)$
5	3	3	0.0299	0.7193	0.05494	0.5229	1.3755
	4	4	0.0096	0.64431	0.0337	0.4358	1.4785
	5	5	0.0076	0.5227	0.0300	0.3479	1.5024
	6	5	0.0027	0.4240	0.0175	0.2858	1.4838
	7	6	0.0113	0.4008	0.0277	0.2600	1.5418
	8	7	0.0117	0.3456	0.02531	0.2196	1.5740
	9	8	0.0016	0.2903	0.0159	0.1906	1.5235
	10	9	0.0093	0.2858	0.0164	0.1817	1.5727
15	3	3	0.0092	0.2450	0.0201	0.1729	1.4172
	4	4	0.0040	0.2137	0.0117	0.1418	1.5068
	5	5	0.0064	0.1663	0.0127	0.1093	1.5220
	6	5	0.0037	0.1436	0.0097	0.0931	1.5424
	7	6	0.0053	0.1337	0.0087	0.0842	1.5866
	8	7	0.0112	0.1139	0.0106	0.0720	1.5827
	9	8	0.001	0.1003	0.0051	0.0646	1.5523
	10	9	0.0067	0.0941	0.0095	0.05865	1.6053
25	3	3	0.0056	0.1467	0.0094	0.0978	1.5007
	4	4	0.0035	0.1295	0.0085	0.0843	1.5362
	5	5	0.0020	0.0979	0.0072	0.0653	1.5004
	6	5	0.0042	0.0825	0.0075	0.0546	1.5110
	7	6	0.0049	0.0816	0.0079	0.0507	1.6095
	8	7	0.0010	0.0667	0.0062	0.0421	1.5855
	9	8	0.0069	0.0588	0.0062	0.03836	1.5318
	10	9	0.0007	0.0576	0.0018	0.0359	1.6037
35	3	3	0.0061	0.1048	0.0094	0.0720	1.4549
	4	4	-0.0009	0.0930	0.0075	0.0602	1.5464
	5	5	0.0015	0.0717	0.0065	0.0470	1.5237
	6	5	0.0034	0.0607	0.0037	0.0392	1.5482
	7	6	0.0016	0.0564	0.0033	0.0355	1.5891
	8	7	0.0031	0.0490	0.0061	0.0304	1.6126
	9	8	0.0028	0.0421	0.0051	0.0272	1.5464
	10	9	0.0017	0.0405	0.0034	0.0256	1.5848

7. MLE UNDER IMPERFECT SAMPLING

In Sections 3, the MLE of the exponential distribution was obtained using RSS-Grouping data under the prefect ranking assumption of RSS and for fixed threshold as well. In practice, however, visual ranking and thresholding may not be perfect. They may contain ranking error and thresholding error due to the informal ranking procedure. So, it is important to study the effects of ranking error on the MLE of RSS-grouping data. In this section, we consider the model of Dell and Clutter (1972) which has the following scenario for ranking errors. Assume that the i^{th} visual score for the i^{th} observation in RSS set is defined as $V_i = X_i + \epsilon_i$, where $\epsilon_1, \ldots, \epsilon_n$ is random sample from $N(0,\ \tau^2)$ and is independent of $X_1, \ldots, X_n$. Then an RSS sample with ranking errors, is obtained according to the following additive model:

1. Obtain $V_i = X_i + \epsilon_i, i = 1, \ldots, n$
2. Rank the V_i in an ascending order $V_{(1)} \leq V_{(2)} \leq \ldots \leq V_{(n)}$ and obtain $X_{[i]}$, the value of X associated with the i^{th} value $V_{(i)}$.
3. Then $X_{[1]}, \ldots, X_{[n]}$ represent an RSS sample with raking errors.

It is clear from Dell and Clutter (1972) model that the random noises $\epsilon_i, i = 1, \ldots, n$, are representing the ranking errors, so the parameter τ reflects the ranking accuracy, i.e., the smaller the value of τ, the more the ranking accuracy and vice versa. Following the ranking errors model in Dell and Clutter (1972), we conduct a simulation study to to assess the effect of the ranking errors on both bias and mean squared error of the MLE. We assume that the experimenter's judgment ranking error for the grouping part of the sample follows a Bernoulli distribution with parameter $\delta \in (0, 1)$, where δ represents the probability that an observation below the threshold T is classified by the experimenter to be above T. Furthermore, the effects of T is also considered in two cases when T is known and when T is unknown. When T is unknown, T_κ the κ^{th} percentile of the population can be used as reasonable proposal for T, where $\kappa \in (0, 1)$. In this case, T will be a parameter and hence can be estimated using the non-grouping part of the RSS-Grouping sample. We restrict our study to the MLE of the parameter of the exponential distribution since it has shown good properties compared to the several estimators introduced in this paper. Specifically, we study the statistical properties of the $\widehat{\theta}_3$ in the presence of ranking errors and when the threshold T is chosen to be T_κ. Under the exponential distribution model, $T = T_\kappa = -\theta log(1 - \kappa)$ and an estimator of T is $\widehat{T} = -\widehat{\theta}_3 \log(1 - \kappa)$. We conduct a simulation study when $\theta = 3$, $\tau = 0.1,\ 0.5,\ 1.0$, $\delta = 0.01,\ 0.05,\ 0.1$ and $\kappa = 0.25,\ 0.50,\ 0.75$. Simulation results (when T is known) are presented in Tables 7 and 8 while for the second case are presented in 9 to 13. The large sample distribution of the MLE under the effect of ranking errors is compared empirically to its fitted normal distribution counterpart for different cases that are displayed in Figures 1 and 2.

TABLE 7. - *Bias and MSE of* $\widehat{\theta}_3$ *for* $\theta = 3,\ T = 1.5,$ *and* $\tau = 0.1,\ 0.5$ *and 1.0 under ranking errors (no ranking errors in the grouped part)*

			$\tau = 0.1$		$\tau = 0.5$		$\tau = 1$	
r	m	j_0	BIAS	MSE	BIAS	MSE	BIAS	MSE
5	3	3	0.0583	0.5368	0.0367	0.5335	-0.0099	0.5406
	4	4	0.0480	0.4437	0.0174	0.4527	-0.0219	0.4772
	5	5	0.0414	0.3464	0.0134	0.3499	-0.0186	0.3590
	6	5	0.0309	0.2762	0.0084	0.2909	-0.0466	0.3210
	7	6	0.0249	0.2524	-0.0026	0.2719	-0.0545	0.3071
	8	7	0.0228	0.2203	-0.0015	0.2304	-0.0618	0.2696
	9	8	0.0193	0.1909	-0.0024	0.2059	-0.0594	0.2415
	10	9	0.0157	0.1785	-0.0074	0.1956	-0.0712	0.2385
15	3	3	0.0166	0.1667	-0.0006	0.1712	-0.0495	0.1750
	4	4	0.0114	0.1429	-0.0083	0.1476	-0.0490	0.1593
	5	5	0.0121	0.0121	-0.0135	0.1159	-0.0654	0.1272
	6	5	0.0090	0.0923	-0.0135	0.0951	-0.0719	0.1085
	7	6	0.0092	0.0836	-0.0167	0.0889	-0.0744	0.1065
	8	7	0.0078	0.0726	-0.0185	0.0758	-0.0798	0.0932
	9	8	0.0066	0.0636	-0.0171	0.0674	-0.0804	0.0842
	10	9	0.0040	0.0588	-0.0212	0.0646	-0.0866	0.0834
25	3	3	0.0090	0.1005	-0.0066	0.1013	-0.0548	0.1077
	4	4	0.0088	0.0850	-0.0142	0.0876	-0.0573	0.0961
	5	5	0.0071	0.0658	-0.0149	0.0684	-0.0711	0.0780
	6	5	0.0052	0.0552	-0.0156	0.0568	-0.0747	0.0675
	7	6	0.0054	0.0506	-0.0215	0.0529	-0.0791	0.0660
	8	7	0.0046	0.0430	-0.0210	0.0463	-0.0837	0.0594
	9	8	0.0040	0.0378	-0.0198	0.0407	-0.0825	0.0535
	10	9	0.0019	0.0354	-0.0231	0.0392	-0.0879	0.0538
35	3	3	0.0068	0.0718	-0.0085	0.0718	-0.0583	0.0780
	4	4	0.0052	0.0605	-0.0184	0.0627	-0.0610	0.0699
	5	5	0.0052	0.0472	-0.0180	0.0494	-0.0727	0.0580
	6	5	0.0033	0.0387	-0.0161	0.0409	-0.0763	0.0496
	7	7	0.0030	0.0356	-0.0206	0.0385	-0.0807	0.0494
	8	7	0.0017	0.0307	-0.0217	0.0329	-0.0855	0.0444
	9	8	0.0013	0.0271	-0.0196	0.0291	-0.0857	0.0407
	10	9	0.0005	0.0251	-0.0251	0.0281	-0.0900	0.0408

TABLE 8. - *Bias and MSE of $\widehat{\theta}_3$ for $\theta = 3$, $T = 1.5$, and $\tau = 0.1$ under ranking errors*

			$\delta = 0.01$		$\delta = 0.05$		$\delta = 0.1$	
r	m	j_0	BIAS	MSE	BIAS	MSE	BIAS	MSE
5	3	3	0.0632	0.5377	0.1082	0.5653	0.1618	0.6067
	4	4	0.0522	0.4394	0.0908	0.4643	0.1445	0.5073
	5	5	0.0421	0.3467	0.0703	0.3620	0.1028	0.3779
	6	5	0.0363	0.2813	0.0592	0.2922	0.0936	0.3087
	7	6	0.0268	0.2585	0.0568	0.2656	0.0788	0.2759
	8	7	0.0216	0.2188	0.0468	0.2284	0.0693	0.2341
	9	8	0.0236	0.1958	0.0426	0.1990	0.0627	0.2064
	10	9	0.0239	0.1815	0.0411	0.1876	0.0591	0.1921
15	3	3	0.0245	0.1686	0.0724	0.1825	0.1261	0.2023
	4	4	0.0238	0.1438	0.0610	0.1527	0.1060	0.1682
	5	5	0.0181	0.1131	0.0471	0.1188	0.0821	0.1277
	6	5	0.0176	0.0930	0.0421	0.0960	0.0707	0.1031
	7	6	0.0139	0.0846	0.0364	0.0875	0.0627	0.0922
	8	7	0.0137	0.0728	0.0301	0.0741	0.0506	0.0781
	9	8	0.0116	0.0635	0.0262	0.0649	0.0474	0.0689
	10	9	0.0102	0.0599	0.0260	0.0610	0.0456	0.0641
25	3	3	0.0189	0.1016	0.0597	0.1087	0.1176	0.1243
	4	4	0.0155	0.0842	0.0531	0.0923	0.0998	0.1026
	5	5	0.0134	0.0665	0.0429	0.0705	0.0774	0.0780
	6	5	0.0095	0.0547	0.0342	0.0568	0.0630	0.0617
	7	6	0.0103	0.0501	0.0320	0.0526	0.0588	0.0564
	8	7	0.0092	0.0425	0.0268	0.0446	0.0492	0.0476
	9	8	0.0080	0.0378	0.0244	0.0391	0.0436	0.0411
	10	9	0.0070	0.0360	0.0221	0.0369	0.0403	0.0392
35	3	3	0.0192	0.0720	0.0587	0.0778	0.1147	0.0918
	4	4	0.0127	0.0610	0.0505	0.0653	0.0973	0.0755
	5	5	0.0100	0.0474	0.0400	0.0508	0.0761	0.0565
	6	5	0.0083	0.0397	0.0329	0.0416	0.0602	0.0452
	7	6	0.0078	0.0359	0.0291	0.0375	0.0567	0.0410
	8	7	0.0070	0.0307	0.0243	0.0324	0.0485	0.0343
	9	8	0.0066	0.0273	0.0230	0.0282	0.0416	0.0299
	10	9	0.0050	0.0253	0.0206	0.0262	0.0401	0.0280

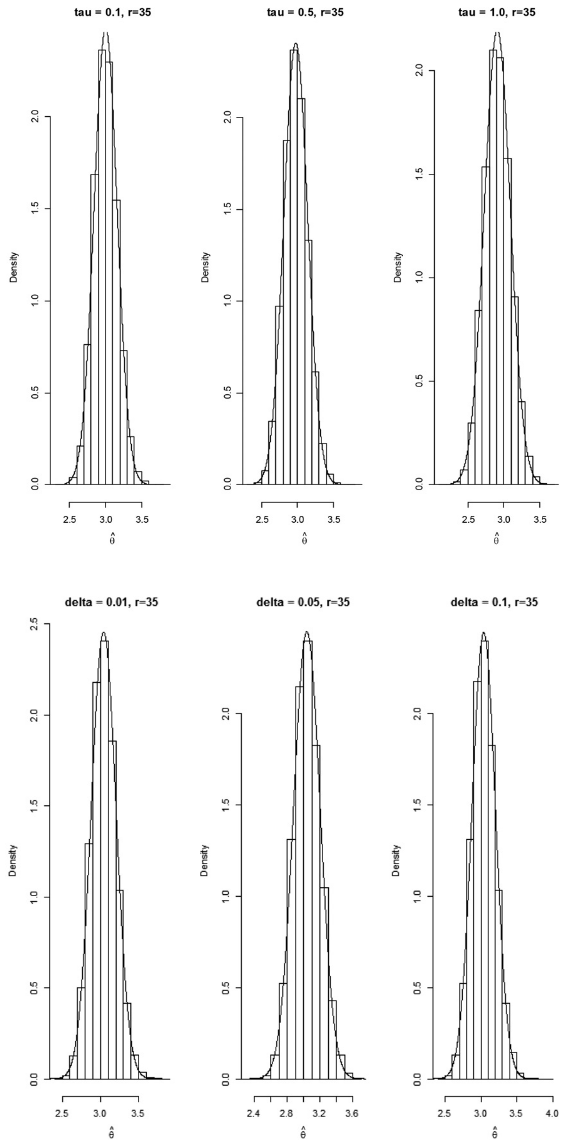

FIGURE 1. - *The empirical distribution function of* $\hat{\theta}_3$ *for* $\theta = 3$*,* $r = 35$*,* $m = 10$*,* $j_0 = 9$ *and different values of* τ *and* δ

From Tables 7 and 8 and Figure 1 we show that the bias of the MLE is decaying to zero and this implies that the MLE is behaved asymptotically unbiased. Also, the MSE is decaying to 0 and this establishes the consistency of the MLE. More over, the ranking errors don't have clear effect on the asymptotic normality of the MLE.

TABLE 9. -*Bias and MSE of* $\widehat{\theta}_3$ *for* $\theta = 3,\ T = 1.5$, *and* $\tau = 0.1$ *under ranking errors*

			$\kappa = 0.25$		$\kappa = 0.5$		$\kappa = 0.75$	
r	m	j_0	BIAS	MSE	BIAS	MSE	BIAS	MSE
5	3	3	0.1369	0.6648	0.1310	0.5449	0.1498	0.5102
	4	4	0.1069	0.5497	0.1018	0.4457	0.1198	0.4124
	5	5	0.0847	0.4172	0.0817	0.3375	0.0947	0.3072
	6	5	0.0739	0.3457	0.0733	0.2775	0.0792	0.2508
	7	6	0.0636	0.3132	0.0592	0.2489	0.0714	0.2269
	8	7	0.0534	0.2604	0.0491	0.2118	0.0625	0.1925
	9	8	0.0493	0.2305	0.0425	0.1851	0.0547	0.1664
	10	9	0.0444	0.2163	0.0391	0.1723	0.0468	0.1535
15	3	3	0.1176	0.2220	0.1147	0.1866	0.1347	0.1779
	4	4	0.0980	0.1852	0.0920	0.1521	0.1078	0.1411
	5	5	0.0796	0.1408	0.0735	0.1164	0.0859	0.1059
	6	5	0.0634	0.1147	0.0604	0.0931	0.0733	0.0863
	7	6	0.0594	0.1059	0.0549	0.0839	0.0652	0.0765
	8	7	0.0491	0.0884	0.0468	0.0710	0.0564	0.0649
	9	8	0.0422	0.0777	0.0409	0.0621	0.0491	0.0559
	10	9	0.0414	0.0736	0.0374	0.0578	0.0457	0.0518
25	3	3	0.1141	0.1385	0.1137	0.1170	0.1303	0.1122
	4	4	0.0930	0.1145	0.0906	0.0936	0.1107	0.0892
	5	5	0.0758	0.0889	0.0702	0.0698	0.0857	0.0666
	6	5	0.0636	0.0705	0.0607	0.0585	0.0716	0.0537
	7	6	0.0562	0.0634	0.0534	0.0510	0.0622	0.0470
	8	7	0.0493	0.0542	0.0466	0.0434	0.0548	0.0398
	9	8	0.0428	0.0473	0.0413	0.0379	0.0472	0.0341
	10	9	0.0394	0.0441	0.0375	0.0353	0.0448	0.0318
35	3	3	0.1123	0.1014	0.1093	0.0863	0.1291	0.0845
	4	4	0.0960	0.0840	0.0899	0.0684	0.1056	0.0661
	5	5	0.0738	0.0637	0.0711	0.0510	0.0844	0.0492
	6	5	0.0613	0.0508	0.0583	0.0419	0.0680	0.0385
	7	6	0.0544	0.0464	0.0533	0.0370	0.0628	0.0348
	8	7	0.0481	0.0391	0.0456	0.0314	0.0530	0.0290
	9	8	0.0411	0.0337	0.0401	0.0276	0.0485	0.0254
	10	9	0.0386	0.0316	0.0371	0.0254	0.0429	0.0232

TABLE 10. -*Bias and MSE of* $\widehat{\theta}_3$ *for* $\theta = 3$ *and* $\tau = 0.1,\ \delta = 0.05$ *under ranking errors*

			$\kappa = 0.25$		$\kappa = 0.5$		$\kappa = 0.75$	
r	m	j_0	BIAS	MSE	BIAS	MSE	BIAS	MSE
5	3	3	0.0738	0.6146	0.0647	0.4989	0.0871	0.4706
	4	4	0.0532	0.5156	0.0523	0.4137	0.0676	0.3821
	5	5	0.0476	0.4018	0.0412	0.3238	0.0532	0.2857
	6	5	0.0426	0.3280	0.0395	0.2660	0.0409	0.2364
	7	6	0.0303	0.2979	0.0269	0.2393	0.0427	0.2107
	8	7	0.0285	0.2513	0.0285	0.2055	0.0309	0.1821
	9	8	0.0264	0.2223	0.0253	0.1797	0.0292	0.1597
	10	9	0.0230	0.2124	0.0238	0.1684	0.0251	0.1469
15	3	3	0.0616	0.2014	0.0510	0.1651	0.0720	0.1551
	4	4	0.0489	0.1711	0.0483	0.1373	0.0535	0.1238
	5	5	0.0382	0.1317	0.0371	0.1061	0.0421	0.0953
	6	5	0.0342	0.1082	0.0317	0.0881	0.0384	0.0792
	7	6	0.0262	0.0991	0.0264	0.0781	0.0309	0.0696
	8	7	0.0263	0.0854	0.0219	0.0678	0.0280	0.0607
	9	8	0.0236	0.0745	0.0207	0.0599	0.0261	0.0530
	10	9	0.0185	0.0691	0.0174	0.0558	0.0238	0.0497
25	3	3	0.0563	0.1201	0.0550	0.1011	0.0672	0.0935
	4	4	0.0464	0.1029	0.0455	0.0843	0.0507	0.0754
	5	5	0.0361	0.0795	0.0361	0.0647	0.04225	0.0582
	6	5	0.0305	0.0656	0.0310	0.0532	0.0343	0.0475
	7	6	0.0269	0.0593	0.0279	0.0479	0.0306	0.0425
	8	7	0.0234	0.0509	0.0226	0.0406	0.0276	0.0366
	9	8	0.0213	0.0446	0.0201	0.0359	0.0237	0.0317
	10	9	0.0211	0.0423	0.0189	0.0332	0.0229	0.0296
35	3	3	0.0587	0.0878	0.0564	0.0732	0.0643	0.0670
	4	4	0.0444	0.0730	0.0457	0.0600	0.0534	0.0538
	5	5	0.0366	0.0568	0.0361	0.0462	0.0439	0.0416
	6	5	0.0303	0.0473	0.0301	0.0377	0.0342	0.0341
	7	6	0.0260	0.0425	0.0255	0.0343	0.0311	0.0305
	8	7	0.0236	0.0368	0.0226	0.0294	0.0257	0.0259
	9	8	0.0201	0.0319	0.0211	0.0257	0.0233	0.0231
	10	9	0.0200	0.0304	0.0173	0.0237	0.0221	0.0212

TABLE 11 - *Bias and MSE of* $\widehat{\theta}_3$ *and* $\tau = 0.1$, $\delta = 0.01$ *under ranking errors*

			$\kappa = 0.25$		$\kappa = 0.5$		$\kappa = 0.75$	
r	m	j_0	BIAS	MSE	BIAS	MSE	BIAS	MSE
5	3	3	0.0236	0.5813	0.0262	0.4765	0.0320	0.4283
	4	4	0.0148	0.4904	0.0157	0.3988	0.0208	0.3539
	5	5	0.0133	0.3811	0.0138	0.3085	0.0201	0.2774
	6	5	0.0168	0.3138	0.0113	0.2566	0.0184	0.2304
	7	6	0.0083	0.2922	0.0086	0.2343	0.0149	0.2060
	8	7	0.0087	0.2530	0.0101	0.2010	0.0113	0.1769
	9	8	0.0060	0.2158	0.0089	0.1768	0.0110	0.1544
	10	9	0.0086	0.2074	0.0069	0.1643	0.0083	0.1428
15	3	3	0.0163	0.1882	0.0156	0.1542	0.0179	0.1404
	4	4	0.0113	0.1616	0.0111	0.1298	0.0161	0.1157
	5	5	0.0092	0.1267	0.0083	0.1023	0.0113	0.0910
	6	5	0.0082	0.1047	0.0057	0.0848	0.0102	0.0747
	7	6	0.0061	0.0960	0.0052	0.0762	0.0077	0.0671
	8	7	0.0058	0.0828	0.0055	0.0654	0.0057	0.0574
	9	8	0.0057	0.0731	0.0046	0.0577	0.0069	0.0516
	10	9	0.0048	0.0689	0.0033	0.0543	0.0054	0.0478
25	3	3	0.0138	0.1132	0.0140	0.0926	0.0141	0.0831
	4	4	0.0070	0.0965	0.0095	0.0778	0.0112	0.0685
	5	5	0.0083	0.0758	0.0085	0.0610	0.0088	0.0539
	6	5	0.0093	0.0752	0.0051	0.0503	0.0090	0.0441
	7	6	0.0052	0.0577	0.0051	0.0461	0.0056	0.0402
	8	7	0.0056	0.0496	0.0051	0.0395	0.0067	0.0345
	9	8	0.0036	0.0436	0.0037	0.0347	0.0053	0.0305
	10	9	0.0033	0.0412	0.0035	0.0325	0.0050	0.0285
35	3	3	0.0131	0.0809	0.0121	0.0663	0.0145	0.0596
	4	4	0.0099	0.0689	0.0068	0.0553	0.0110	0.0495
	5	5	0.0078	0.0541	0.0073	0.0436	0.0098	0.0384
	6	5	0.0069	0.0447	0.0065	0.0360	0.0081	0.0319
	7	6	0.0064	0.0413	0.0039	0.0326	0.0066	0.0289
	8	7	0.0061	0.0353	0.0050	0.0283	0.0057	0.0248
	9	8	0.0040	0.0311	0.0046	0.0251	0.0049	0.0218
	10	9	0.0039	0.0293	0.0037	0.0231	0.0046	0.0203

TABLE 12. - *Bias and MSE of* $\widehat{\theta}_3$ *for* $\theta = 3$ *and* $\tau = 0.5$, $\delta = 0.05$ *under ranking errors*

			$\kappa = 0.25$		$\kappa = 0.5$		$\kappa = 0.75$	
r	m	j_0	BIAS	MSE	BIAS	MSE	BIAS	MSE
5	3	3	0.0795	0.6126	0.0739	0.5007	0.087	0.4682
	4	4	0.0584	0.516	0.0510	0.4141	0.0666	0.3760
	5	5	0.0466	0.398	0.0441	0.3251	0.0508	0.2900
	6	5	0.0412	0.3277	0.0372	0.2634	0.0463	0.2374
	7	6	0.0349	0.3014	0.0324	0.2375	0.0367	0.2114
	8	7	0.0291	0.2546	0.0254	0.2016	0.0332	0.1816
	9	8	0.0288	0.2231	0.0258	0.1790	0.0318	0.1594
	10	9	0.0262	0.2114	0.0238	0.1676	0.0247	0.14701
15	3	3	0.0650	0.2019	0.0582	0.1657	0.0700	0.1531
	4	4	0.0493	0.1701	0.0482	0.1389	0.0567	0.1247
	5	5	0.0398	0.1302	0.0382	0.1076	0.0467	0.0965
	6	5	0.0324	0.1077	0.0320	0.0878	0.0369	0.0781
	7	6	0.0295	0.0995	0.0270	0.0797	0.0335	0.0704
	8	7	0.0255	0.0842	0.0245	0.0675	0.0290	0.0602
	9	8	0.0228	0.0741	0.0220	0.0593	0.0252	0.0532
	10	9	0.0201	0.0697	0.0200	0.0555	0.0235	0.0493
25	3	3	0.0577	0.1200	0.0580	0.1015	0.0679	0.0936
	4	4	0.0519	0.1043	0.0461	0.0832	0.0558	0.0760
	5	5	0.0386	0.0794	0.0369	0.0645	0.0427	0.0583
	6	5	0.0330	0.0647	0.0308	0.0529	0.0370	0.0477
	7	6	0.0298	0.0603	0.0264	0.0473	0.0322	0.0426
	8	7	0.0254	0.0510	0.0229	0.0404	0.0288	0.0364
	9	8	0.0214	0.0448	0.0207	0.0356	0.0248	0.0322
	10	9	0.0202	0.0419	0.0199	0.0338	0.0229	0.0294
35	3	3	0.0570	0.0875	0.0565	0.0732	0.0656	0.0680
	4	4	0.0453	0.0738	0.0453	0.0599	0.0545	0.0547
	5	5	0.0374	0.0569	0.0368	0.0463	0.0427	0.0416
	6	5	0.0310	0.0471	0.0303	0.0378	0.0357	0.0346
	7	6	0.0284	0.0429	0.0267	0.0342	0.0311	0.0306
	8	7	0.0241	0.0366	0.0244	0.0293	0.0279	0.0262
	9	8	0.0208	0.0318	0.0198	0.0258	0.02423	0.0228
	10	9	0.0208	0.0298	0.0192	0.0238	0.0226	0.0213

TABELE 13. - *Bias and MSE of $\widehat{\theta}_3$ for $\theta = 3$ and $\tau = 1.0$, $\delta = 0.05$ under ranking errors*

			$\kappa = 0.25$		$\kappa = 0.5$		$\kappa = 0.75$	
r	m	j_0	BIAS	MSE	BIAS	MSE	BIAS	MSE
5	3	3	-0.0041	0.6440	0.0015	0.5284	0.0271	0.4830
	4	4	-0.0316	0.5862	-0.0166	0.4644	0.0050	0.4121
	5	5	-0.0525	0.4630	-0.0371	0.3674	-0.0150	0.3221
	6	5	-0.0563	0.3883	-0.0422	0.3055	-0.0248	0.2676
	7	6	-0.0707	0.3863	-0.0531	0.2990	-0.0326	0.2519
	8	7	-0.0763	0.3391	-0.0617	0.2589	-0.0420	0.2189
	9	8	-0.0808	0.2962	-0.0651	0.2285	-0.0460	0.1954
	10	9	-0.0915	0.3062	-0.0729	0.2306	-0.0513	0.1927
15	3	3	-0.0160	0.2104	-0.0055	0.1714	0.0137	0.1555
	4	4	-0.0307	0.1932	-0.0217	0.1519	-0.001	0.1325
	5	5	-0.0556	0.1550	-0.0379	0.1216	-0.0207	0.1054
	6	5	-0.0600	0.1301	-0.0433	0.1021	-0.0297	0.0881
	7	6	-0.0724	0.1316	-0.0527	0.0997	-0.0390	0.0839
	8	7	-0.0784	0.1140	-0.0608	0.0876	-0.0418	0.0731
	9	8	-0.0812	0.1031	-0.0620	0.0777	-0.0465	0.0657
	10	9	-0.0885	0.1059	-0.0691	0.0789	-0.0527	0.0652
25	3	3	-0.0185	0.1259	-0.0054	0.1022	0.0102	0.0928
	4	4	-0.0327	0.1154	-0.0196	0.0901	-0.0015	0.0793
	5	5	-0.0541	0.0938	-0.0368	0.0729	-0.0214	0.0631
	6	5	-0.0621	0.0799	-0.0465	0.0622	-0.0305	0.0532
	7	6	-0.0729	0.0798	-0.0535	0.0607	-0.0375	0.0503
	8	7	-0.0795	0.0714	-0.0599	0.0535	-0.0438	0.0445
	9	8	-0.0808	0.0641	-0.0611	0.0479	-0.0471	0.0401
	10	9	-0.0893	0.0666	-0.0670	0.0483	-0.0530	0.0399
35	3	3	-0.0186	0.0901	-0.0074	0.0735	0.0094	0.0660
	4	4	-0.0334	0.0827	-0.0183	0.0650	-0.0024	0.0564
	5	5	-0.0544	0.0679	-0.0388	0.0524	-0.0221	0.0449
	6	5	-0.0632	0.0580	-0.0467	0.0450	-0.0303	0.0379
	7	6	-0.0723	0.0588	-0.0535	0.0440	-0.0371	0.0364
	8	7	-0.0786	0.0525	-0.0604	0.0391	-0.0444	0.0323
	9	8	-0.0805	0.0476	-0.0625	0.0355	-0.0482	0.0294
	10	9	-0.0893	0.0499	-0.0678	0.0360	-0.0533	0.0293

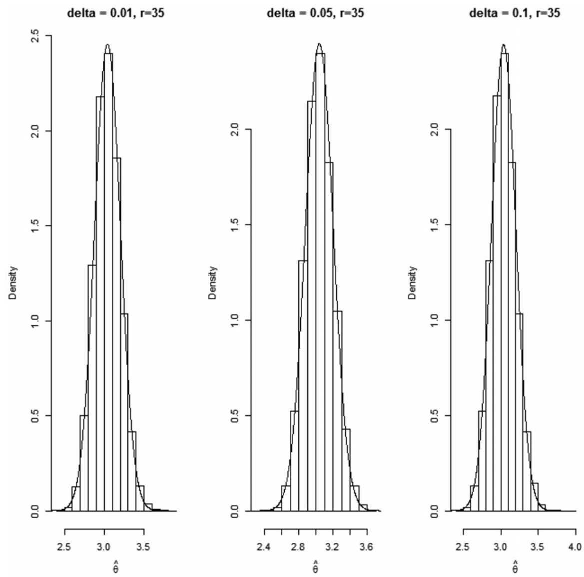

FIGURE 2. - *The empirical distribution function of* $\hat{\theta}_3$ *for* $\theta = 3$*,* $r = 35$*,* $m = 10$*,* $j_0 = 9$ *and different values of* κ

Tables 6 and 7 show that absolute bias of the MLE and its MSE are decreasing functions of r. On the other hand the absolute bias and the MSE of the MLE are increasing functions of τ, i.e., the more the errors in ranking, the less the accuracy of the MLE. By comparing the results in Table 8 (the last two columns, $\delta = 0.1$) with the results in Table 9, we can see that there is slightly small effect of ranking errors on both biases and MSEs of MLE. Additionally, Table 8 shows that absolute bias has no clear pattern as a function of r while the MSE is a decreasing function of the sample size(r) and κ.

Tables 9 to 11 demonstrate that absolute bias and MSE of the MLE are decreasing functions of δ. Tables 10, 12, and 13 illustrate that the absolute bias and the MSE are decreasing as functions of τ. 5. In spite of the fact that there is a slightly effect of ranking errors on bias and MSE of the MLE, it is clear from Figures 1 and 2 that it does not have a significant effect for the large sample distribution of the MLE.

8. CONCLUSIONS

In this paper, a new sampling technique is proposed that contains more information than RSS. The proposed sampling technique can be implemented using the concept of visual grouping of population units with respect to a fixed threshold and the standard ranked set sample. We have implemented this technique to estimate the mean of the exponential distribution using maximum likelihood method. In particular, the MLE under the best informative RSS-Grouping is derived and compared to several estimators that were derived under various sampling techniques. A simulation study was conducted to compare among these estimators and the MLE under our proposed scheme performed better. Our sampling scheme is then extended to study its performance under imperfect sampling, imperfect ranking of visual grouping, and when the threshold is unknown. For this purpose, we have considered the ranking errors model given by Dell and Clutter (1972), assumed that the experimenter's judgment ranking error for the grouping part of the sample follows a Bernoulli distribution with parameter $\delta \in (0, 1)$, and the thereshold T was considered as the κ^{th} percentile of the exponential distribution, where $\kappa \in (0, 1)$. We have conducted a simulation study to assess the MLE under these assumptions and simulation results were showed that the MLE was behaved asymptotically unbiased. The MLE under this scenario tended to be at least as efficient as the MLE under RSS regardless of raking errorsand the estimation of the threshold had slightly effects on the sampling distribution of the MLE. Although, we have applied the RSS-Grouping sampling to the exponential distribution, the RSS-Grouping is general and can be applied to other distributions. We propose the following modifications to the proposed technique as open problems for future work:

1. Instead of quantifying the data $Z = \left\{Z_{ij};\ i = 1,\ 2,\ ..,m,\ j = 1,\ 2,\ 3,..,r\right\}$ from parallel independent samples, we may consider the following sampling scheme. From the i^{th} set, $i = 1,\ 2, \ldots, m$, in the RSS, quantify the following

   (i) $Y_i = X_{i(i)}$ and Z_i the number of units with characteristic exceeding a fixed known threshold T, i.e., $Z_i = \sum_{j=1,\, j\neq i}^{m} I(X_{i(j)} > T)$.

   (ii) Repeat the step (i) r times to get a sample of size $n = mr$.

   It can be noticed that for each i, the two random variables Y_i and Z_i are not independent. Although the dependence of Y_i and Z_i adds difficulty to the analysis, it can reduce the number of units needed for visual ranking, i.e., from each set we get two observations namely Y_i and Z_i.

2. Different thresholds $T_1,\ \ldots,\ T_m$ with $T_1 <\ \ldots <\ T_m$ can be used for each set, i.e., for the i^{th} set the threshold T_i can be used for visual grouping. In this case the population is stratified via these known thresholds. For example if $m = 2$, let Z and W be such that Z is the number of units, through a sample of size m, that are detected to have characteristic below the threshold T_1. Hence, $Z \sim Bin(m,\ F(T_1;\theta))$. Similarly, if W denotes the number of units that have characteristic exceeding T_2 in another sample of size m independent from the first sample, then $W \sim Bin\big(m,\ \big(m,\ 1 - F(T_2;\theta)\big)$. Let $Z_1, \ldots, Z_r$ and $W_1, \ldots, W_r$ be copies of Z and W, respectively. Then the likelihood of $Z_1, \ldots, Z_r, W_1, \ldots, W_r$ is

$$L(\theta; Z_1, \ldots, Z_r, W_1, \ldots, W_r) = \prod_{i=1}^{r} \binom{m}{Z_i} F(T_1;\theta)^{Z_i} (1 - F(T_1;\theta))^{m-Z_i} \times$$

$$\prod_{i=1}^{r} \binom{m}{W_i} F(T_2;\theta)^{m-W_i} (1 - F(T_2;\theta))^{W_i}.$$

So this likelihood can be used to make inference about θ. Also several statistical techniques such those in Al-Rawwash *et al.* (2010), Alodat, Alodat, Al-Rawwash, and Alodat (2009a), and Alodat, Al-Rawwash and Nawajah (2009b, 2010) can be used to make statistical inference under the modified RSS-Grouping.

ACKNOWLEDGEMENTS

The authors thank the editor and two anonymous referees for their helpful comments and suggestions that led to improvements in the manuscript.

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