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ANALYSIS OF STRUCTURAL BREAK IN VAR(K) TIME SERIES MODEL: A BAYESIAN APPROACH

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SUMMARY

Vector autoregressive (VAR) model is the most popular modeling tool in macroeconomics. This study considers a Bayesian framework for VAR(k) model with a structural break in the mean. The structural change problem in VAR is of theoretical and practical importance in reference to the economic time series data. The main motivation of the study is to identify the impact of the break in the series and estimate the model parameters in the presence of the break considering appropriate prior assumptions. A simulation study and empirical analysis of the net asset value of national pension schemes for different fund managers have been carried out to justify the proposed mechanism.

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1. INTRODUCTION

Vector autoregressive (VAR) time series models are the model in which current observation is modeled based on past observations and it is an initial point for the analysis of dynamic econometric problems. The proposal of the VAR approach was first presented at a conference on business cycle research by the Federal Reserve Bank of Minneapolis. These are commonly used on the dynamic data to study the interaction among the economic variables of interest (Sims, 1972). Later, Sargent and Sims (1977) showed VARs as the general form of conventional macro models. They proceeded to estimate the parameters of the VAR model and simplify the VAR in order to identify and compare the data-based model in the form of a particular macro model. VARs are an essential tool for empirical macro-

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economic research. The dynamic interdependency between variables representing the Granger causality hypothesis (Granger, 1969) was captured by these models as shown by Sims (1980). Further, VARs can be shown to approximate other models such as Vector Autoregressive Moving Average models (Lutkepohl and Poskitt, 1996).

The initial study of VAR with the Bayesian approach was explored by Litterman (1979) and more specifically VAR estimation under the Bayesian framework was firstly supported by Litterman (1980) to the over-fitting problem in the classical approach. Later on, Litterman (1986) established Bayesian VARs as benchmark models for economic forecasting. Doan, Litterman and Sims (1984) used the parameter shrinkage in order to check the accuracy of the models. Giannone and Reichlin (2009) simultaneously with Alessi and Banbura (2009) were recommended to estimate Bayesian VARs with a large cross-section rather than using shrinkage properties for VAR estimation. Alessi and Banbura (2009) concluded that the Bayesian approach in VAR is the global structure that will be able to yield exact forecasts. Bayesian VARs are successfully used in macroeconomic forecasting with a large number of variables.

In the present study, VARs are dealt considering with the problem of structural change under Bayesian inference. The problem of structural change has an extensive concern in time series and econometrics. There are so many studies on structural change problems available in the literature under the classical approach (Chow, 1960; Quandt 1960; Andrews, 1993; Stock, 1994; Bai and Perron, 1998). In recent years, many univariate statistics have been developed to test for the presence of structural breaks in stationary and nonstationary time series. Perron (1989) analyzed the Nelson-Plosser data set and found that many series are stationary around segmented means. Perron and Vogelsang (1992) and Perron (1997) proposed a class of test statistics for two different forms of structural break which allows the changes in both level and trend. Structural change in many of Plosser time series data was confirmed by Vogelsang (1997) and Chu and White (1992) using direct tests for shifts in trend. If there are breaks in the univariate series, it seems natural that the breaks should also appear in a multivariate system. Keeping this view in concern Ng and Vogelsang (2002) discussed the VAR model in the presence of a shift in mean and explored the consequences of unstable means for estimation, inference, and forecasting. Chib (1998) developed a Bayesian approach for the estimation of multiple change points via the Monte Carlo Markov Chain (MCMC) algorithm. He provided a comprehensive study for evaluation and model selection for the autoregressive (AR) model. Koop and Potter (2007) considered regime-specific parameters of a Markov-switching model and assumed a Poisson hierarchical prior that allows dependence on regime-specific parameters. Sugita (2008) analyzed the multiple structural breaks in the vector autoregressive model under the Bayesian framework and examined the performance in detecting the number of breaks and estimating their location using the Bayes factor. Koop and Potter (2009) discussed the elicitation of priors in change point models under consideration of unknown multiple breakpoints in regression and time series models. Jochmann, Koop, and Strachan (2010) investigated the performance of Bayesian forecasting of the VAR model in the account of structural breaks in the VAR parameters.

Preuss, Puchstein, and Dette (2015) introduced a new nonparametric procedure referred to MuBreD procedure for detection and identification of the position of multiple breakpoints in the multivariate time series. MuBreD is based on a comparison of estimated spectral distribution on different segments of the observed time series. Bacchiocchi and Fanelli (2015) developed a rank condition for local identification of SVARs, where both the error covariance matrix and the structural parameters are allowed to change across volatility regimes. Cho and Fryzlewicz (2015) introduced a sparsified binary segmentation algorithm for the detection of multiple change-points in high dimensional time series. This algorithm aggregates the cumulative sum statistics by adding only those that pass a certain threshold. Safikhani and Shojaie (2022) considered a three-stage procedure to identify the number of breaks and their location in a high-dimensional piecewise vector autoregressive model. They also provided consistent estimates for both structural change points and model parameters. Maheu and Song (2018) proposed a Bayesian approach in the multivariate vector autoregressive model for estimation and forecasting of multiple structural breaks. Kurita and Nielsen (2019) considered a partial co-integrated vector autoregressive model subject to structural breaks in linear trend and constant. The asymptotic distribution of the proposed likelihood-based test statistics for co-integrating rank is also introduced by them. Gao, Yang and Yang (2020) proposed a Lasso with OLS method to estimate the number and location of change points in a stationary vector autoregressive model and further estimate parameters of different regions.

In the present work, we have enhanced the problem of structural break through Bayesian inference. Throughout the paper, our focus is on the Bayesian VAR model considering a break in mean and derived the Posterior Odds Ratio for testing the presence of a break in the model. This type of derivation was proposed in a text by Zellner and Montmarquette (1971) for multiple linear models. First, the posterior odds ratio is derived under appropriate prior assumptions for testing the presence of structural breakpoint in the multiple series and estimating the model parameters using the conditional posterior distribution. A simulation series is generated from the derived model to record the performance of Bayesian inference. For empirical analysis, we have applied the time series of daily Net Asset Value (NAV) and taken all three schemes simultaneously under both tiers. Tier I and Tier II of different fund managers for finding and estimating the parameter in presence of break point.

National Pension Scheme (NPS) was started by the government of India to begin with a system that facilitates all citizens of India to secure their future when they are not in a position to work due to old age. The main concern of any retirement plan like a pension is to become a source of income after retirement. Pension schemes are designed as a tool to provide post-employment benefits on the basis of contribution during the employment age. Mainly pensions are designed into two groups contributory and non-contributory. NPS is a contributory scheme in which 10% of gross income excluding perks is contributed by employees in the pension funds and the fund is invested in the market. In India, the NPS fund is regulated by Pension Fund Regulatory Development Authority (PFRDA). PFRDA recognizes the Bank and other institutions that may participate in NPS fund activities. As there are several banks and insurance companies which are permitted by PFRDA to become the fund managers, they are investing the fund as per the structure given in Figure 1.

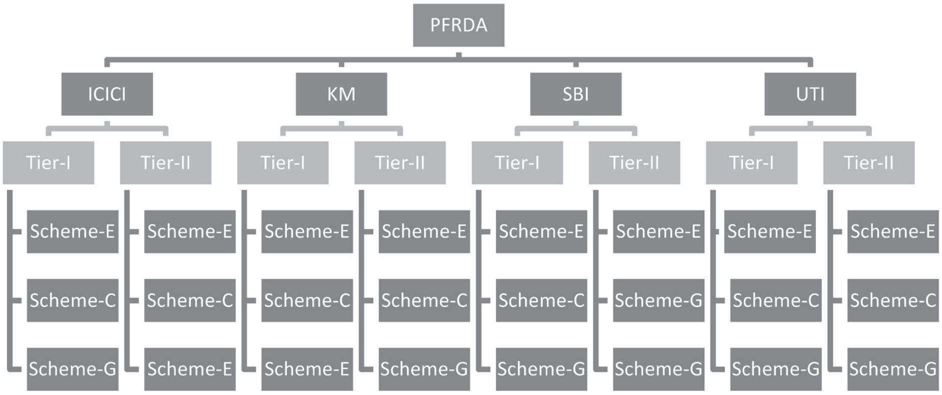

FIGURE 1. - *PFRDA structure for managing NPS fund activities*

From the PFRDA structure, it is clear that NPS funds are managed by multiple banks through two mechanisms; first not allowed withdrawal before retirement named Tier-I, and second allowed withdrawal named Tier-II. Both funds are invested in equity market (E), fixed return (C), and government bonds (G). Kumar, Chaturvedi, and Afifa (2017) analyzed data of NPS for testing the stationarity of NAV series using the unit root hypothesis and concluded that series are trend stationery. Kumar, Afifa, and Chaturvedi (2018) again tested the stationarity of NAV series through unit root testing for panel data time series model with time effect and found that series are trend stationary. The motive behind the choice of the NPS dataset for the problem of Structural Break in the VAR(k) Time Series Model is to know the presence of the break in mean may change the structure of the model or not and after identifying the breakpoint, the adequacy of the model has been checked based on NAV series of NPS.

2. VECTOR AUTOREGRESSIVE TIME SERIES MODEL WITH STRUCTURAL BREAK

Let us consider the VAR Time series model with order k contaminated by structural break presence at a single time point T_b as:

$$Y_t = \begin{cases} \mu_1 + A_{11}Y_{t-1} + A_{12}Y_{t-2} + \cdots + A_{1k}Y_{t-k} + \eta_t; & for\ \ t = 1,2,\ldots,T_b \\ \mu_2 + A_{21}Y_{t-1} + A_{22}Y_{t-2} + \cdots + A_{2k}Y_{t-k} + \eta_t; & for\ \ t = T_b + 1,...,T \end{cases} \tag{1}$$

where N is a number of variables under study, Y_t and η_t are $1xN$, $A_{i,j}$is $N \times N$ .The disturbances η_t are unobservable random variables with $E(\eta_t\ ) = 0$ and $Var(\eta_t) = \Sigma$ an identity matrix and assume that the error term η_t is normally distributed with mean set to a vector of zeros and covariance matrix Σ as $\eta_t \sim N_N(0, \Sigma)$.The coefficients $A_{i,j}$ and variance- covariance matrices Σ_i, $i = 1,2$, are all unknown. Now, we can write the model (1) in matrix notation as follows:

$$Y_t = \begin{cases} X_t'\theta_1 + \eta_t & for\ \ t = 1,2,\ldots,T_b \\ X_t'\theta_2 + \eta_t & for\ \ t = T_b + 1,\ldots,T \end{cases} \tag{2}$$

where $X_t' = \left(1, Y_{t-1}', Y_{t-2}', \ldots, Y_{t-k}'\right)$ and $\theta_i = \left(\mu_i, A_{i1}, A_{i2}, \ldots, A_{ik}\right)'$.

Model (2) is a vector autoregressive of order *k* time series model when structural break presence at breakpoint T_b S-VAR(k). Model (2) may be represented considering no break in mean $(\theta_1 = \theta_2)$. Then, S-VAR(k) model reduces to

$$Y_t = X_t'\theta + \eta_t \quad for \;\; t = 1,2,\ldots,T \tag{3}$$

The purpose of this paper is to make inferences about the structural breaks in the vector autoregressive model under the Bayesian framework. First of all, a test hypothesis is proposed to examine the presence of breakpoint in the parameter and then parameters are estimated by derived posterior distribution. The proposed research problem is also verified through empirical analysis of the National Pension Scheme.

3. BAYESIAN INFERENCE

The main intention of any time series model is to construct a forecasting mechanism to predict the future trend by utilizing available information. In order to achieve the proposed objective, one may be initially interested to draw significant inferences about the structure of the model by considering some estimation, testing, and model selection procedures. So, the objective of the present section is to analyze the Bayesian estimation and testing procedure for the derived model to handle a certain particular situation.

3.1 *Prior assumptions*

The selection of the prior distribution for the model parameters of the VAR model can sustain by an extensive range of opinions. The prior distributions which are assumed in the study are closely related to the distributions assumed in some of the papers (Kadiyala and Karlsson, 1997; Del Negro and Schorfheide, 2004; Villani, 2009; Banbura, Giannone and Reichlin, 2010; Koop, 2013). We consider a basic prior distribution that enables analytical derivation of the posterior distribution and, thus, fast computations. The natural-conjugate prior distribution is a matric-variate normal conditional prior distribution of θ given Σ and an inverse wishart marginal prior distribution for Σ. Let us assume the following prior distributions for the parameters used in the models are as:

$$\theta_1|\Sigma \sim MN\left(\overline{\theta}_1, \Sigma, V\right)$$

$$\theta_2|\Sigma \sim MN\left(\overline{\theta}_2, \Sigma, V\right)$$

$$\Sigma \sim IW_N\left(S, \nu\right)$$

The joint prior probability of all parameters for S-VAR model is

$$\pi(\theta_1, \theta_2, \Sigma) = \frac{(2\pi)^{-N(1+Nk)} |\Sigma|^{-(1+Nk)} |V|^{-N} |s|^{\frac{\nu}{2}}}{|\Sigma|^{\frac{\nu+N+1}{2}} 2^{\frac{\nu N}{2}} \Gamma\left(\frac{\nu}{2}\right)} \exp\left[-\frac{1}{2} tr\left\{\Sigma^{-1}\left(\left(\theta_1 - \overline{\theta}_1\right)' V^{-1}\left(\theta_1 - \overline{\theta}_1\right) + \left(\theta_2 - \overline{\theta}_2\right)' V^{-1}\left(\theta_2 - \overline{\theta}_2\right) + S\right)\right\}\right] \tag{4}$$

3.2 *Posterior probabilities*

The main motive behind the present study is to test the presence of break-in autoregressive coefficient with intercept term. In the Bayesian approach, hypothesis testing is acquired by using the posterior odds ratio (POR). POR is derived with the help of posterior probability under the null and alternative hypotheses. Let us assume the following notations to obtain the posterior probability:

$P_1 = XX' + V^{-1}$ $\quad$ $P_2 = XY + V^{-1}\overline{\theta}$

$P_3 = Y'Y + \overline{\theta}'V^{-1}\overline{\theta} + S$ $\quad$ $\tilde{\theta}_{H_1} = P_1^{-1}P_2$

$P = P_3 - \tilde{\theta}'_{H_1} P_1 \tilde{\theta}_{H_1}$ $\quad$ $L_1 = \left(V^{-1} + X_1 X_1'\right)$

$L_2 = X_1 Y_1 + V^{-1}\overline{\theta}_1$ $\quad$ $L_3 = Y_1'Y_1 + \overline{\theta}_1' V^{-1}\overline{\theta}_1$

$\tilde{\theta}_{H_2} = L_1^{-1} L_2$ $\quad$ $L = L_3 - \tilde{\theta}'_{H_2} L_1 \tilde{\theta}_{H_2}$

$M_1 = \left(V^{-1} + X_2 X_2'\right)$ $\quad$ $M_2 = X_2 Y_2 + V^{-1}\overline{\theta}_2$

$M_3 = Y_2'Y_2 + \overline{\theta}_2' V^{-1}\overline{\theta}_2$ $\quad$ $\tilde{\theta}_{bH_2} = M_1^{-1} M_2$

$M = M_3 - \tilde{\theta}'_{bH_2} M_1 \tilde{\theta}_{bH_2}$

The posterior probability under H_1, when model is considering no break, is given by:

$$P(Y|H_1) = \frac{|V|^{-\frac{N}{2}} \; |s|^{\frac{\nu}{2}} \; 2^{\frac{NT}{2}} \; \Gamma\left(\frac{\nu+T}{2}\right)}{(2\pi)^{\frac{NT}{2}} \; \Gamma\left(\frac{\nu}{2}\right) \left|P_1^{-1}\right|^{\frac{-N}{2}} \; |P|^{\frac{\nu+T}{2}}} \tag{5}$$

The posterior probability under H_2, when model is considering break in mean at known break point T_b, is given by:

$$P(Y|H_2) = \frac{|V|^{-N}|s|^{\frac{\nu}{2}}2^{\frac{T\ N}{2}}\Gamma\left(\frac{\nu+T}{2}\right)}{(2\pi)^{\frac{NT}{2}}\Gamma\left(\frac{\nu}{2}\right)\left|L_1^{-1}\right|^{\frac{-N}{2}}\left|M_1^{-1}\right|^{\frac{-N}{2}}|B|^{\frac{\nu+T}{2}}} \tag{6}$$

3.3 *Posterior odds ratio*

In the Bayesian testing procedure, the comparison of two different hypotheses can be made by using the posterior odds ratio (β_{01}) and it is given as:

$$\beta_{01} = \frac{P(H_0|y)}{P(H_1|y)} = O(H_0)\frac{P(y|H_0)}{P(y|\mid H_1)} = \frac{p_0}{1-p_0}\frac{P(y|H_0)}{P(y|H_1)}$$

For testing the null hypothesis$H_1 : \theta_1 = \theta_2$ *i.e.* model is considering no break in the series against the alternative hypothesis $H_2 : \theta_1 \neq \theta_2$ *i.e.*, under the consideration of break in mean, expressed as below:

$$\beta_{01} = \frac{p_0}{1-p_0}\frac{|V|^{\frac{N}{2}}\left|L_1^{-1}\right|^{\frac{-N}{2}}\left|M_1^{-1}\right|^{\frac{-N}{2}}|B|^{\frac{\nu+T}{2}}}{\left|P_1^{-1}\right|^{\frac{-N}{2}}|P|^{\frac{\nu+T}{2}}} \tag{7}$$

3.4 *Conditional posterior distribution*

Parameter estimation discusses the process by using sample data to estimate the parameters of the selected model *i.e.,* it made inferences about unknown quantities of interest related to a real data set. In frequentist statistics, one uses observed data to construct a point estimate for each model parameter. The MLE and bias-adjusted version of the MLE are examples of this. In Bayesian statistics parameter estimation involves placing a probability distribution over model parameters. In fact, there is no conceptual difference between parameter estimation (inferences about θ) and density estimation (inferences about future y) in Bayesian statistics. Bayes estimator of parametric function is calculated by conditional posterior probability. The likelihood function of the model with $\Theta = \{\theta_1, \theta_2, \Sigma\}$is:

$$L(Y|\Theta) = \frac{1}{(2\pi)^{\frac{NT}{2}}|\Sigma|^{\frac{T}{2}}} \exp\left[-\frac{1}{2}tr\left\{\Sigma^{-1}\left(\left(Y_1 - X_1'\theta_1\right)'\left(Y_1 - X_1'\theta_1\right) + \left(Y_2 - X_2'\theta_2\right)'\left(Y_2 - X_2'\theta_2\right)\right)\right\}\right]$$

Operating the likelihood function with joint prior distribution (6), we have derived conditional posterior distribution and get the marginal distribution of every parameter which may depend on other parameters. For computing the conditional posterior distribution for θ_1, θ_2 and Σ, we get the expression as:

$$\Pi(\theta_1|\theta_2,\Sigma,y) \propto \exp\left[-\frac{1}{2}tr\left\{\Sigma^{-1}(\theta_1 - B_3)' B_1(\theta_1 - B_3)\right\}\right]$$

$$\Pi(\theta_2|\theta_1,\Sigma,y) \propto \exp\left[-\frac{1}{2}tr\left\{\Sigma^{-1}(\theta_1 - F_3)' F_1(\theta_1 - F_3)\right\}\right]$$

$$\Pi(\Sigma|\theta_1,\theta_2,y) \propto \frac{1}{|\Sigma|^{\frac{2(1+Nk)+(T+\nu+N+1)}{2}}}\exp\left[-\frac{1}{2}tr\left\{\Sigma^{-1}(G)\right\}\right]$$

where

$B_1 = X_1X_1' + V^{-1}$; $\quad B_2 = X_1Y_1 + V^{-1}\overline{\theta}_1$; $\quad B_1^{-1}B_2 = B_3$

$F_1 = X_2X_2' + V^{-1}$; $\quad F_2 = X_2Y_2 + V^{-1}\overline{\theta}_2$; $\quad F_1^{-1}F_2 = F_3$

$$G = (Y_1 - X_1'\theta_1)'(Y_1 - X_1'\theta_1) + (Y_2 - X_2\theta_2)'(Y_2 - X_2'\theta_2) + S$$
$$+V^{-1}\left\{(\theta_1 - \overline{\theta}_1)(\theta_1 - \overline{\theta}_1) + (\theta_2 - \overline{\theta}_2)'(\theta_2 - \overline{\theta}_2)\right\}$$

We can easily see that the marginal distribution of θ_1, θ_2 , and Σ has closed-form solution and conditional posterior distribution of θ_1 and θ_2 has matric-variate normal distribution and Σ has an inverse Wishart distribution.

4. SIMULATION STUDY

The importance of statistics is to draw a significant inference for a given model using adequate statistical techniques. Simulation is a flexible methodology to analyze the behavior of a proposed study and compares the best estimate. Here, we have studied the behavior of the VAR model in presence of structural break on the basis of the simulated samples with varying sample sizes and different combinations of the VAR coefficient. For this purpose, we have simulated the series of different sizes T= (80, 100, 120) with different positions of breakpoint *(T/4, T/2, 3T/4)*. The initial value of the response series is Y =(10, 20) considering the number of variables $N=2$. The true value of the VAR coefficient is described as

$$\theta_1 = \begin{pmatrix} 2 & 3 \\ 0.15 & 0.3 \\ 0.1 & 0.2 \end{pmatrix}; \quad \theta_2 = \begin{pmatrix} 5 & 7 \\ 0.2 & 0.1 \\ 0.2 & 0.15 \end{pmatrix}; \quad \Sigma = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 2 \end{pmatrix}$$

First, we obtained the estimators of parameters of the proposed model for each generated sample, then reported the absolute bias and the corresponding mean square

TABLE 1. - *AE, MSE and AB of the estimator θ_1 with varying T and T_B*

T	T_B	θ_1	2	3	0.15	0.30	0.10	0.20
80	$T/4$	AE	2.1054	3.1377	0.0846	0.2746	0.1104	0.1943
		MSE	0.2421	0.5247	0.1369	0.5263	0.1795	0.1285
		AB	0.3916	0.5684	0.2941	0.5788	0.3088	0.2793
	$T/2$	AE	2.0619	3.0750	0.1230	0.3131	0.1031	0.1837
		MSE	0.1322	0.2956	0.0627	0.2520	0.0716	0.0575
		AB	0.2846	0.4316	0.1990	0.3912	0.2006	0.1896
	$3T/4$	AE	2.0377	3.0605	0.1308	0.2848	0.1055	0.1915
		MSE	0.0887	0.1920	0.0368	0.1556	0.0417	0.0371
		AB	0.2350	0.3435	0.1531	0.3100	0.1582	0.1516
100	$T/4$	AE	2.1120	3.1374	0.0987	0.2586	0.1155	0.1952
		MSE	0.1914	0.4261	0.1038	0.3836	0.0994	0.0901
		AB	0.3408	0.5115	0.2548	0.4881	0.2417	0.2331
	$T/2$	AE	2.0705	3.0964	0.1197	0.2745	0.1035	0.1959
		MSE	0.1046	0.2353	0.0475	0.1811	0.0496	0.0433
		AB	0.2551	0.3810	0.1760	0.3330	0.1715	0.1654
	$3T/4$	AE	2.0425	3.0745	0.1458	0.3034	0.0938	0.1822
		MSE	0.0809	0.1711	0.0271	0.1133	0.0300	0.0291
		AB	0.2248	0.3285	0.1286	0.2651	0.1335	0.1368
120	$T/4$	AE	2.0631	3.0955	0.1019	0.2673	0.1146	0.2006
		MSE	0.1673	0.3814	0.0845	0.3256	0.0862	0.0675
		AB	0.3226	0.4874	0.2321	0.4550	0.2290	0.2028
	$T/2$	AE	2.0624	3.0657	0.1269	0.2755	0.1126	0.1923
		MSE	0.0929	0.1918	0.0379	0.1504	0.0418	0.0345
		AB	0.2408	0.3495	0.1543	0.3043	0.1598	0.1465
	$3T/4$	AE	2.0442	3.0637	0.1403	0.3021	0.0997	0.1891
		MSE	0.0653	0.1404	0.0243	0.0919	0.0214	0.0236
		AB	0.1968	0.2939	0.1237	0.2400	0.1133	0.1217

errors of Bayes estimators of the model parameters. All results are based on 5000 replications. For the different sizes of the series with varying break locations, average estimates (AE), absolute bias (AB), and mean square error (MSE) of the estimators of the parameters have been summarized in Tables 1-3.

From Tables 1-3, it is concluded that average estimates are near to the actual value of the parameter that can be observed using MSE and AB. As the size of the series increases location of the breakpoint is changed from near to the bottom of the series, and MSE and AB for θ_1 are decreased. And we got reversed results for parameter θ_2 *i.e.,* MSE and AB are increased as the change in location of the break point from near to bottom of the series while similar results are obtained as the size of the series is increased. From the tables shown above, it is also interpreted that change in location parameter does not provide any pattern for parameters of variance-covariance matrix because proposed model considers only break in mean. After estimating the parameters of the proposed model, testing has been done for the change in mean. Table 4 provides the POR value for different sizes of series with varying breakpoints and different combinations of mean parameters. Table 4 shows that the POR value is less than one for each location of breakpoint and series size, i.e., reject the null hypothesis and

TABLE 2. - *AE, MSE and AB of the estimator θ_2 with varying T and T_B*

T	T_B	θ_2	5	7	0.20	0.10	0.20	0.15
80	$T/4$	AE	5.1881	7.2835	0.1738	0.0857	0.2047	0.1365
		MSE	0.5618	1.1313	0.0246	0.5761	0.0367	0.0473
		AB	0.5836	0.8432	0.1266	0.5650	0.1486	0.1751
	$T/2$	AE	5.2181	7.3428	0.1714	0.0632	0.2014	0.1314
		MSE	0.7318	1.5350	0.0331	0.6548	0.0481	0.0630
		AB	0.6719	0.9877	0.1430	0.5868	0.1680	0.1975
	$3T/4$	AE	5.2582	7.4480	0.1709	0.0983	0.2081	0.0914
		MSE	1.1440	2.4056	0.0595	1.3145	0.0836	0.1368
		AB	0.8444	1.2119	0.1896	0.7594	0.2199	0.2971
100	$T/4$	AE	5.1108	7.1605	0.1844	0.1109	0.1975	0.1422
		MSE	0.4093	0.8663	0.0190	0.4473	0.0295	0.0358
		AB	0.5090	0.7283	0.1105	0.4837	0.1294	0.1511
	$T/2$	AE	5.2145	7.3064	0.1625	0.0589	0.2080	0.1526
		MSE	0.6814	1.4265	0.0267	0.5282	0.0380	0.0490
		AB	0.6464	0.9483	0.1303	0.5318	0.1561	0.1761
	$3T/4$	AE	5.3692	7.5722	0.1522	0.0829	0.2034	0.1314
		MSE	1.0943	2.4486	0.0534	0.7933	0.0689	0.0956
		AB	0.8316	1.2372	0.1823	0.6632	0.2055	0.2484
120	$T/4$	AE	5.1053	7.1505	0.1832	0.0390	0.2131	0.1478
		MSE	0.3856	0.8520	0.0176	0.2898	0.0195	0.0289
		AB	0.4872	0.7308	0.1068	0.4006	0.1092	0.1362
	$T/2$	AE	5.1825	7.2857	0.1718	0.0609	0.2056	0.1381
		MSE	0.5565	1.1384	0.0250	0.4256	0.0307	0.0444
		AB	0.5908	0.8248	0.1264	0.4880	0.1374	0.1691
	$3T/4$	AE	5.2611	7.3540	0.1655	0.0884	0.2049	0.1133
		MSE	0.9767	2.0043	0.0445	0.6191	0.0488	0.0790
		AB	0.7716	1.1141	0.1671	0.5907	0.1751	0.2220

conclude that series contains a break in the mean term. Thus, the simulation study correctly generates the sample series from the proposed model.

5. EMPIRICAL ANALYSIS

We studied the NPS funds for estimation and tested the presence of structural breakpoint in the series of schemes under both tiers of different fund managers using the Vector Autoregressive time series Model. For analysis purpose, we have taken three banks namely ICICI, SBI, and Kotak Mahindra and time series recorded daily basis of NAV data for the period February 01, 2010 to December 31, 2017. For appropriate interpretation of the data, we converted the daily series of the individual bank into a monthly average and tested the presence of breakpoint using the derived theorem. Breaks present in the data can distort statistical inference of the model parameters and may lead to the wrong inference. Thus, it is necessary to account for or deal with possible breaks in the data set before making

TABLE 3. - *AE, MSE and AB of the estimator Σ with varying T and T_B*

T	T_B	Σ	1	0.5	0.5	3
80	$T/4$	AE	0.8807	0.3701	0.3701	2.8119
		MSE	0.1573	0.0709	0.0709	1.1891
		AB	0.2978	0.2169	0.2169	0.9399
	$T/2$	AE	0.8637	0.3735	0.3735	2.7437
		MSE	0.1421	0.0741	0.0741	1.1470
		AB	0.2836	0.2245	0.2245	0.9490
	$3T/4$	AE	0.8359	0.3778	0.3778	2.6006
		MSE	0.2341	0.0665	0.0665	1.1835
		AB	0.2990	0.2106	0.2106	0.9510
100	$T/4$	AE	0.8628	0.3557	0.3557	2.8374
		MSE	0.1432	0.0749	0.0749	1.1511
		AB	0.2821	0.2128	0.2128	0.9296
	$T/2$	AE	0.8144	0.3979	0.3979	2.7112
		MSE	0.1640	0.0591	0.0591	1.1213
		AB	0.2825	0.1942	0.1942	0.9013
	$3T/4$	AE	0.7837	0.3848	0.3848	2.6003
		MSE	0.1510	0.0593	0.0593	1.1902
		AB	0.2867	0.1993	0.1993	0.9440
120	$T/4$	AE	0.7763	0.3750	0.3750	2.6018
		MSE	0.0938	0.0613	0.0613	1.1830
		AB	0.2741	0.2008	0.2008	0.9265
	$T/2$	AE	0.7693	0.3805	0.3805	2.6497
		MSE	0.1240	0.0586	0.0586	1.1390
		AB	0.2767	0.1938	0.1938	0.9283
	$3T/4$	AE	0.7394	0.3829	0.3829	2.5021
		MSE	0.1162	0.0529	0.0529	1.1079
		AB	0.2756	0.1818	0.1818	0.9051

TABLE 4. - *Posterior odds ratio with varying T and T_B*

θ_1		(2,3)	(3,5)	(5,7)
θ_2		(5,7)	(3,5)	(2,3)
T	*TB*	*POR*	*POR*	*POR*
80	$T/4$	1.33E-23	3.82E-19	8.53E-21
	$T/2$	1.81E-27	1.39E-21	3.40E-25
	$3T/4$	3.39E-32	8.53E-29	8.87E-30
100	$T/4$	1.87E-23	4.71E-21	2.48E-26
	$T/2$	1.32E-30	5.10E-24	7.02E-28
	$3T/4$	4.99E-39	4.95E-30	2.50E-34
120	$T/4$	4.61E-25	1.69E-22	1.84E-26
	$T/2$	6.47E-31	5.82E-28	6.53E-33
	$3T/4$	1.68E-41	1.07E-32	1.29E-38

any inference for analysis. The present section tested the presence of structural breaks and how it affects the parameters of the model under the Bayesian framework through the real data of NPS.

For estimation and testing purposes, firstly structural break is identified in the NAV series of NPS through a classical approach using the R package “strucchange” under the command “breakpoints” (Zeileis, Leisch, Hornik, Kleiber, 2002) . It is detected the breakpoint for individual series of schemes E, C, and G under both tiers of different banks. We have considered the intersection point of individual schemes of E, C and G for both tiers including different banks under study as a breakpoint for our analysis. The date-wise time periods recorded in the scheme of banks for both tiers are listed in Table 5. The logical reason for these sudden break points occurred in due to local or individual economic crisis, interest, and benefit within the schemes under different banks. These structural changes vary from scheme to scheme due to individual factors affecting domestically on these fund managers. Therefore, we have tested our model considering the given breakpoint listed in Table 5. The model has been explored by fitting the NAV series of NPS and testing the presence of the break in mean hypothesis under the Bayesian framework using the derived POR. The calculated POR for banks under study are recorded in Table 5.

Table 5 shows that if the break is present in the mean only, the posterior odds ratio value is too small to reject the null hypothesis i.e., no break is present in the mean. The presence of break in means to influence the time series data and maybe change the structure of the model. For the data used in the present study, we also say that all series are shifted by level component and this is due to market pension policy. The estimated parameter in the frequentist framework is sometimes unreliable due to the presence of abnormal observations. To overcome this problem, it is necessary to get a better decision about unknown parameters in the presence of a structural break. To justify our theoretical results of estimation, we have estimated the parameters with the help of frequentist estimation and compared the estimators from the Bayesian approach with the classical approach. After identifying the breakpoint which is listed in Table 5, the adequacy of the model has been checked based on the NAV series of NPS. Bayesian estimates of the real data set for the VAR(k) model which considers a break in mean are summarized in Table 6.

6. CONCLUSIONS

In the present paper, the posterior odds ratio is derived for identifying the presence of the break in vector autoregressive time series model considering the break in mean. The Bayesian estimator is provided to estimate the model parameters in consideration of the break in mean. For theoretical justification, simulation and empirical studies are conducted and got the model is more efficient when the series is having a break in mean. This work may be extended for the case of break-in both mean and variance-covariance matrix and panel-VAR model.

TABLE 5. - *Possible date-wise break points and POR in the set schemes E, C and G under both tier*

Banks	Tier	Breakmonth	Break-point	POR
SBI (May,2009 to Dec,2016)	II	April, 2012	36	2.35E-27
ICICI (May,2009 to Dec,2016)	I	May,2010	13	6.20E-15
KM (Dec,2009 to Dec,2016)	I	April, 2012	36	8.31E-34

TABLE 6. - *Bayes estimates of parameter based on real data set*

	Banks		
	SBI TIER II	ICICI TIER I	KM TIER I
θ_1	$\begin{bmatrix} -1.035 & -0.115 & 0.549 \\ 0.397 & 0.002 & -0.006 \\ -0.529 & 0.484 & 0.161 \\ 0.712 & 0.029 & 0.301 \end{bmatrix}$	$\begin{bmatrix} -12.764 & 1.290 & 3.424 \\ 0.154 & 0.052 & 0.051 \\ -0.401 & 0.505 & 0.151 \\ 2.044 & -0.184 & -0.042 \end{bmatrix}$	$\begin{bmatrix} 0.658 & 0.389 & 1.171 \\ 0.404 & -0.017 & -0.010 \\ 0.016 & 0.593 & 0.290 \\ 0.027 & -0.114 & 0.092 \end{bmatrix}$
θ_2	$\begin{bmatrix} 0.073 & -0.207 & -0.390 \\ 0.482 & 0.011 & 0.012 \\ 0.281 & 0.603 & 0.166 \\ -0.277 & -0.100 & 0.345 \end{bmatrix}$	$\begin{bmatrix} 0.435 & 0.015 & -0.006 \\ 0.487 & 0.006 & 0.008 \\ 0.209 & 0.527 & 0.043 \\ -0.248 & -0.033 & 0.447 \end{bmatrix}$	$\begin{bmatrix} 0.070 & -0.208 & -0.392 \\ 0.482 & 0.011 & 0.012 \\ 0.282 & 0.604 & 0.167 \\ -0.278 & -0.100 & 0.344 \end{bmatrix}$
Σ	$\begin{bmatrix} 3.9204 & 6.9813 & -10.3310 \\ 6.9813 & 86.3068 & -95.4427 \\ -10.3310 & -95.4427 & 107.7962 \end{bmatrix}$	$\begin{bmatrix} 65.2441 & 9.7571 & -81.7943 \\ 9.7571 & 3.8129 & -54.2301 \\ -81.7943 & -54.2301 & 151.3139 \end{bmatrix}$	$\begin{bmatrix} 4.919 & -2.1781 & 2.0953 \\ -2.1781 & 110.7701 & -114.1320 \\ 2.0953 & -114.1320 & 117.7970 \end{bmatrix}$

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APPENDIX

A1. POSTERIOR PROBABILITY UNDER H_1

Under this hypothesis, model containing no break in mean and variance and likelihood function is given by:

$$L(\theta,\Sigma|Y) = \frac{1}{(2\pi)^{\frac{NT}{2}}|\Sigma|^{\frac{T}{2}}}\exp\left[-\frac{1}{2}tr\left\{\Sigma^{-1}\left(Y - X'\theta\right)'\left(Y - X'\theta\right)\right\}\right]$$

Combining the likelihood function with the prior distributions, the posterior probability is:

$$P(Y|H_1) = \int_{\Re_{N\times N}}\int_{\Re^+_{N(1+Nk)}} L(\theta,\Sigma \mid Y)\pi(\theta)\pi(\Sigma)d\theta d\Sigma$$

$$= \int_{\Re_{N\times N}}\int_{\Re^+_{N(1+Nk)}} \frac{(2\pi)^{\frac{-N(1+Nk)}{2}}|\Sigma|^{\frac{-(1+Nk)}{2}}|V|^{-\frac{N}{2}}|s|^{\frac{\nu}{2}}}{(2\pi)^{\frac{NT}{2}}|\Sigma|^{\frac{\nu+T+N+1}{2}}2^{\frac{\nu N}{2}}\Gamma\left(\frac{\nu}{2}\right)}$$
$$\exp\left[-\frac{1}{2}tr\left[\Sigma^{-1}\left\{\left(Y - X'\theta\right)'\left(Y - X'\theta\right) + \left(\theta - \bar{\theta}\right)'V^{-1}\left(\theta - \bar{\theta}\right) + S\right\}\right]\right]d\theta d\Sigma$$

$$= \int_{\Re_{N\times N}}\int_{\Re^+_{N(1+Nk)}} \frac{(2\pi)^{\frac{-N(1+Nk)}{2}}|\Sigma|^{\frac{-(1+Nk)}{2}}|V|^{-\frac{N}{2}}|s|^{\frac{\nu}{2}}}{(2\pi)^{\frac{NT}{2}}|\Sigma|^{\frac{\nu+T+N+1}{2}}2^{\frac{\nu N}{2}}\Gamma\left(\frac{\nu}{2}\right)}$$
$$\exp\left[-\frac{1}{2}tr\left[\Sigma^{-1}\left\{\theta'\left(XX' + V^{-1}\right)\theta - 2\theta'\left(XY + V^{-1}\bar{\theta}\right) + (Y'Y + \bar{\theta}'V^{-1}\bar{\theta} + S)\right\}\right]\right]d\theta d\Sigma$$

Using the notation given in Section 3.2

$$= \int_{\Re_{N\times N}}\int_{\Re^+_{N(1+Nk)}} \frac{(2\pi)^{\frac{-N(1+Nk)}{2}}|\Sigma|^{\frac{-(1+Nk)}{2}}|V|^{-\frac{N}{2}}|s|^{\frac{\nu}{2}}}{(2\pi)^{\frac{NT}{2}}|\Sigma|^{\frac{\nu+T+N+1}{2}}2^{\frac{\nu N}{2}}\Gamma\left(\frac{\nu}{2}\right)}$$
$$\exp\left[-\frac{1}{2}tr\left\{\Sigma^{-1}\left(\theta'P_1\theta - 2\theta'P_2 + P_3\right)\right\}\right]d\theta d\Sigma$$

$$= \int_{\Re_{N\times N}} \frac{(2\pi)^{\frac{-N(1+Nk)}{2}}|\Sigma|^{\frac{-(1+Nk)}{2}}|V|^{-\frac{N}{2}}|s|^{\frac{\nu}{2}}}{(2\pi)^{\frac{NT}{2}}|\Sigma|^{\frac{\nu+T+N+1}{2}}2^{\frac{\nu N}{2}}\Gamma\left(\frac{\nu}{2}\right)}\exp\left[-\frac{1}{2}tr\left\{\Sigma^{-1}\left(P_3 - \tilde{\theta}'_{H_1}P^{-1}\tilde{\theta}_{H_1}\right)\right\}\right]d\Sigma$$
$$\int_{\Re^+_{N(1+Nk)}}\exp\left[-\frac{1}{2}tr\left[\Sigma^{-1}\left\{\left(\theta - \tilde{\theta}_{H_1}\right)'P_1\left(\theta - \tilde{\theta}_{H_1}\right)\right\}\right]\right]d\theta$$

$$= \int_{\Re_{N\times N}} \frac{|V|^{-\frac{N}{2}}|s|^{\frac{\nu}{2}}}{(2\pi)^{\frac{NT}{2}}|\Sigma|^{\frac{\nu+T+N+1}{2}}2^{\frac{\nu N}{2}}\Gamma\left(\frac{\nu}{2}\right)\left|P_1^{-1}\right|^{\frac{-N}{2}}}\exp-\frac{1}{2}tr\left[\sum^{-1}P\right]d\Sigma$$

$$= \frac{|V|^{-\frac{N}{2}}|s|^{\frac{\nu}{2}}}{(2\pi)^{\frac{NT}{2}}2^{\frac{\nu N}{2}}\Gamma\left(\frac{\nu}{2}\right)\left|P_1^{-1}\right|^{\frac{-N}{2}}}\frac{2^{\frac{(T+\nu)N}{2}}\Gamma\left(\frac{\nu+T}{2}\right)}{|P|^{\frac{\nu+T}{2}}}$$

$$= \frac{|V|^{-\frac{N}{2}}|s|^{\frac{\nu}{2}}2^{\frac{NT}{2}}\Gamma\left(\frac{\nu+T}{2}\right)}{(2\pi)^{\frac{NT}{2}}\Gamma\left(\frac{\nu}{2}\right)\left|P_1^{-1}\right|^{\frac{-N}{2}}|P|^{\frac{\nu+T}{2}}}$$

A2. POSTERIOR PROBABILITY UNDER H_2

Under this hypothesis, the likelihood function containing break in mean is given by:

$$L(\theta_1,\theta_2,\Sigma|Y) = \frac{1}{(2\pi)^{\frac{NT}{2}}|\Sigma|^{\frac{T}{2}}} \exp\left[-\frac{1}{2}tr\left\{\sum^{-1}\left(\left(Y_1 - X_1'\theta_1\right)'\left(Y_1 - X_1'\theta_1\right) + \left(Y_2 - X_2'\theta_2\right)'\left(Y_2 - X_2'\theta_2\right)\right)\right\}\right]$$

Combining the likelihood function with the prior distributions, the posterior probability is:

$$P(Y|H_2) = \int_{\Re_{N\times N}}\int_{\Re^+_{N(1+Nk)}}\int_{\Re^+_{N(1+Nk)}} L\left(\theta_2,\Sigma_1,\Sigma_2|Y\right)\pi(\theta_1)\pi(\theta_2)\pi(\Sigma)\,d\theta_1\,d\theta_2\,d\Sigma$$

$$= \int_{\Re_{N\times N}}\int_{\Re^+_{N(1+Nk)}}\int_{\Re^+_{N(1+Nk)}} \frac{(2\pi)^{-N(1+Nk)}|\Sigma|^{-(1+Nk)}|V|^{-N}|s|^{\frac{\nu}{2}}}{(2\pi)^{\frac{NT}{2}}|\Sigma|^{\frac{\nu+T+N+1}{2}}2^{\frac{\nu N}{2}}\Gamma\left(\frac{\nu}{2}\right)}\exp\left[-\frac{1}{2}tr\left(S\Sigma^{-1}\right)\right]$$
$$\exp\left[-\frac{1}{2}tr\left\{\sum^{-1}\left(\theta_2-\overline{\theta}_2\right)'V^{-1}\left(\theta_2-\overline{\theta}_2\right)+\sum^{-1}\left(Y_2-X_2'\theta_2\right)'\left(Y_2-X_2'\theta_2\right)\right\}\right]$$
$$\exp\left[-\frac{1}{2}tr\left\{\sum^{-1}\left(\theta_1-\overline{\theta}_1\right)'V^{-1}\left(\theta_1-\overline{\theta}_1\right)+\sum^{-1}\left(Y_1-X_1'\theta_1\right)'\left(Y_1-X_1'\theta_1\right)\right\}\right]d\theta_1 d\theta_2 d\Sigma$$

$$= \int_{\Re_{N\times N}}\int_{\Re^+_{N(1+Nk)}}\int_{\Re^+_{N(1+Nk)}} \frac{(2\pi)^{-N(1+Nk)}|\Sigma|^{-(1+Nk)}|V|^{-N}|s|^{\frac{\nu}{2}}}{(2\pi)^{\frac{NT}{2}}|\Sigma|^{\frac{\nu+T+N+1}{2}}2^{\frac{\nu N}{2}}\Gamma\left(\frac{\nu}{2}\right)}\exp\left[-\frac{1}{2}tr\left(S\Sigma^{-1}\right)\right]$$
$$\exp\left[-\frac{1}{2}tr\left\{\sum^{-1}\left(\theta_2-\overline{\theta}_2\right)'V^{-1}\left(\theta_2-\overline{\theta}_2\right)+\sum^{-1}\left(Y_2-X_2'\theta_2\right)'\left(Y_2-X_2'\theta_2\right)\right\}\right]$$
$$\exp-\frac{1}{2}tr\left[\Sigma^{-1}\left\{\theta_1'\left(V^{-1}+X_1X_1'\right)\theta_1-2\theta_1'\left(V^{-1}\overline{\theta}_1+X_1Y_1\right)+(\overline{\theta}_1'V^{-1}\overline{\theta}_1+Y_1'Y_1)\right\}\right]d\theta_1 d\theta_2 d\Sigma$$

Using the notation given in Section 3.2, we can write the above equation as follows:

$$= \int_{\Re_{N\times N}} \int_{\Re^+_{N(1+Nk)}} \int_{\Re^+_{N(1+Nk)}} \frac{(2\pi)^{-N(1+Nk)}|\Sigma|^{-(1+Nk)}|V|^{-N}|s|^{\frac{\nu}{2}}}{(2\pi)^{\frac{NT}{2}}|\Sigma|^{\frac{\nu+T+N+1}{2}} 2^{\frac{\nu N}{2}}\Gamma\left(\frac{\nu}{2}\right)}$$

$$\exp-\frac{1}{2}tr\left[\Sigma^{-1}L_1\left\{\theta_1'\theta_1 - 2\theta_1'\tilde{\theta}_{H_3} + L_1^{-1}L_3\right\}\right]$$

$$\exp\left[-\frac{1}{2}tr\left\{\Sigma^{-1}\left(\theta_2-\overline{\theta}_2\right)'V^{-1}\left(\theta_2-\overline{\theta}_2\right)+\Sigma^{-1}\left(Y_2-X_2'\theta_2\right)'\left(Y_2-X_2'\theta_2\right)\right\}\right]$$

$$\exp\left[-\frac{1}{2}tr\left(S\Sigma^{-1}\right)\right]d\theta_1 d\theta_2 d\Sigma$$

$$= \int_{\Re_{N\times N}} \int_{\Re^+_{N(1+Nk)}} \frac{(2\pi)^{-N(1+Nk)}|\Sigma|^{-(1+Nk)}|V|^{-N}|s|^{\frac{\nu}{2}}}{(2\pi)^{\frac{NT}{2}}|\Sigma|^{\frac{\nu+T+N+1}{2}} 2^{\frac{\nu N}{2}}\Gamma\left(\frac{\nu}{2}\right)} \exp\left[-\frac{1}{2}tr\left(S\Sigma^{-1}\right)\right]$$

$$\exp\left[-\frac{1}{2}tr\left\{\Sigma^{-1}\left(L_3-\tilde{\theta}_{H_3}'L_1\tilde{\theta}_{H_3}\right)\right\}\right]$$

$$\exp\left[-\frac{1}{2}tr\left\{\Sigma^{-1}\left(\theta_2-\overline{\theta}_2\right)'V^{-1}\left(\theta_2-\overline{\theta}_2\right)+\Sigma^{-1}\left(Y_2-X_2'\theta_2\right)'\left(Y_2-X_2'\theta_2\right)\right\}\right]$$

$$\int_{\Re^+_{N(1+Nk)}} \exp-\frac{1}{2}tr\left[\Sigma^{-1}\left\{(\theta_1-\tilde{\theta}_{H_3})'L_1(\theta_1-\tilde{\theta}_{H_3})\right\}\right]d\theta_1 d\theta_2 d\Sigma$$

$$= \int_{\Re_{N\times N}} \int_{\Re^+_{N(1+Nk)}} \frac{(2\pi)^{\frac{-N(1+Nk)}{2}}|\Sigma|^{\frac{-(1+Nk)}{2}}|V|^{-N}|s|^{\frac{\nu}{2}}}{(2\pi)^{\frac{NT}{2}}|\Sigma|^{\frac{\nu+T+N+1}{2}} 2^{\frac{\nu \quad N}{2}}\Gamma\left(\frac{\nu}{2}\right)\left|L_1^{-1}\right|^{\frac{-N}{2}}}$$

$$\exp\left[-\frac{1}{2}tr\left\{\Sigma^{-1}\left(L_3-\tilde{\theta}_{H_3}{}'L_1\tilde{\theta}_{H_3}+S\right)\right\}\right]$$

$$\exp\left[-\frac{1}{2}tr\left\{\Sigma^{-1}\left(\theta_2-\overline{\theta}_2\right)'V^{-1}\left(\theta_2-\overline{\theta}_2\right)+\Sigma^{-1}(Y_2-X_2{}'\theta_2)'(Y_2-X_2{}'\theta_2)\right\}\right]d\theta_2 d\Sigma$$

$$= \int_{\Re_{N\times N}} \int_{\Re^+_{N(1+Nk)}} \frac{(2\pi)^{\frac{-N(1+Nk)}{2}}|\Sigma|^{\frac{-(1+Nk)}{2}}|V|^{-N}|s|^{\frac{\nu}{2}}}{(2\pi)^{\frac{NT}{2}}|\Sigma|^{\frac{\nu+T+N+1}{2}} 2^{\frac{\nu \quad N}{2}}\Gamma\left(\frac{\nu}{2}\right)\left|L_1^{-1}\right|^{\frac{-N}{2}}}$$

$$\exp\left[-\frac{1}{2}tr\left\{\Sigma^{-1}\left(L_3-\tilde{\theta}_{H_2}{}'L_1\tilde{\theta}_{H_2}+S\right)\right\}\right]$$

$$\exp\left[-\frac{1}{2}tr\left\{\Sigma^{-1}\left(\theta_2{}'\left(V^{-1}+X_2X_2{}'\right)\theta_2-2\theta_2{}'\left(V^{-1}\overline{\theta}_2+X_2Y_2\right)+(\overline{\theta}_2{}'V^{-1}\overline{\theta}_2+Y_2{}'Y_2)\right)\right\}\right]d\theta_2 d\Sigma$$

Using the notation given in Section 3.2, write the equation as:

$$
= \int\limits_{\Re_{N\times N}} \frac{(2\pi)^{\frac{-N(1+Nk)}{2}} |\Sigma|^{\frac{-(1+Nk)}{2}} |V|^{-N} |s|^{\frac{\nu}{2}}}{(2\pi)^{\frac{NT}{2}} |\Sigma|^{\frac{\nu+T+N+1}{2}} 2^{\frac{\nu N}{2}} \Gamma\left(\frac{\nu}{2}\right) \left|L_1^{-1}\right|^{\frac{-N}{2}}}
$$

$$
\exp - \frac{1}{2} tr\left\{\Sigma^{-1}\left(L_3 - \tilde{\theta}_{H_2}{}' L_1 \tilde{\theta}_{H_2} + S\right)\right\}
$$

$$
\int\limits_{\Re^{+}_{N(1+Nk)}} \exp - \frac{1}{2} tr\left[\Sigma^{-1}\left\{(\theta_2 - \tilde{\theta}_{bH_2})' M_1 (\theta_2 - \tilde{\theta}_{bH_2})\right\}\right]
$$

$$
\exp - \frac{1}{2} tr\left\{\Sigma^{-1}\left(M_3 - \tilde{\theta}_{bH_2}{}' M_1 \tilde{\theta}_{bH_2}\right)\right\} d\theta_2 d\Sigma
$$

$$
= \frac{|V|^{-N} |s|^{\frac{\nu}{2}}}{(2\pi)^{\frac{NT}{2}} 2^{\frac{\nu N}{2}} \Gamma\left(\frac{\nu}{2}\right) \left|L_1^{-1}\right|^{\frac{-N}{2}} \left|M_1^{-1}\right|^{\frac{-N}{2}}} \int\limits_{\Re_{N\times N}} \frac{1}{|\Sigma|^{\frac{\nu+T+N+1}{2}}}
$$

$$
\exp\left[-\frac{1}{2} tr\left\{\Sigma^{-1}\left(M_3 - \tilde{\theta}_{bH_2}{}' M_1 \tilde{\theta}_{bH_2} + L_3 - \tilde{\theta}_{H_2}{}' L_1 \tilde{\theta}_{H_2} + S\right)\right\}\right] d\Sigma
$$

$$
= \frac{|V|^{-N} |s|^{\frac{\nu}{2}} \; 2^{\frac{(\nu+T)N}{2}} \Gamma\left(\frac{\nu+T}{2}\right)}{(2\pi)^{\frac{NT}{2}} 2^{\frac{\nu N}{2}} \Gamma\left(\frac{\nu}{2}\right) \left|L_1^{-1}\right|^{\frac{-N}{2}} \left|M_1^{-1}\right|^{\frac{-N}{2}} |B|^{\frac{\nu+T}{2}}}
$$

$$
= \frac{|V|^{-N} |s|^{\frac{\nu}{2}} 2^{\frac{NT}{2}} \Gamma\left(\frac{\nu+T}{2}\right)}{(2\pi)^{\frac{NT}{2}} \Gamma\left(\frac{\nu}{2}\right) \left|L_1^{-1}\right|^{\frac{-N}{2}} \left|M_1^{-1}\right|^{\frac{-N}{2}} |B|^{\frac{\nu+T}{2}}}
$$